A NOTE ON THE EQUIVALENCE OF ALGEBRAS OF DEGREE TWO*

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The simplest type of normal simple algebra over any non-modular field F is the cyclic algebra of degree two (order four)

$$(1)\quad Q(\alpha, \beta) = (1, i, j, ij), ji = -ij, i^2 = \alpha \neq 0, j^2 = \beta \neq 0,$$

(α and β in F), the so-called generalized quaternion algebra over F. Of great importance in the theory of linear algebras is the problem of finding conditions that two given normal simple algebras of the same degree shall be equivalent. But this problem has not, as yet, been explicitly solved even for the above simplest case of algebras of degree two except when F is an algebraic field.† The purpose of this brief note is to give a simplification of my own previous results for rational algebras of degree two and thereby to give simple explicit conditions that any two generalized quaternion algebras over any non-modular field F shall be equivalent.

We consider an algebra $Q(\alpha, \beta)$. A quantity x in $Q(\alpha, \beta)$ but not in F has the property $x^2 = \gamma$ in F if and only if

$$(2)\quad x = \xi_1 i + (\xi_2 + \xi_3 i)j, \; x^2 = \gamma = \xi_1^2\alpha + \xi_2^2\beta - \xi_3^2\alpha\beta.$$

Suppose first that another algebra $Q(\gamma, \delta)$ has the property $\gamma = \xi_1^2\alpha$ for ξ_1 in F. Then, as is well known,‡ we have the following lemma.

LEMMA. *If $\gamma = \xi_1^2\alpha$, then $Q(\alpha, \beta)$ is equivalent to $Q(\gamma, \delta)$ if and only if*

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* Presented to the Society, February 25, 1933.

† See this Bulletin, vol. 36 (1930), pp. 535–540, for algebras of degree two over any algebraic number field, and Hasse's arithmetic invariant theory in the Transactions of this Society, vol. 34 (1932), for algebras of degree n over any algebraic number field. Hasse's conditions of course have no meaning for the case we are considering.

‡ The case $n = 2$ of Hasse's Theorem (2.12) (loc. cit. p. 173) if $\alpha \neq \epsilon^2$ for any ϵ of F. If $\alpha = \epsilon^2$, then both of the above algebras are total matric algebras and are equivalent for any β and δ. But also the equation $\delta = (\xi_4^2 - \xi_5^2\alpha)\beta \neq 0$ has the solution $2\xi_4 = (1 + \delta\beta^{-1})$, $2\xi_5\epsilon = (1 - \delta\beta^{-1})$.

$$\delta = (\xi_4^2 - \xi_5^2\alpha)\beta\,,$$

for ξ_4 *and* ξ_5 *in* F.

Let next $Q(\alpha, \beta)$ and $Q(\gamma, \delta)$ be equivalent. Since then there must exist an x in $Q(\alpha,\beta)$ for which $x^2=\gamma$, we must have equation $(2)_2$ for ξ_1, ξ_2, ξ_3 not all zero and in F. If $\xi_2=\xi_3=0$, then our equivalence conditions are given in the Lemma above. Otherwise

$$y = \xi_3\alpha j + \xi_2 ij \neq 0, \tag{3}$$

and I have proved* that

$$Q(\alpha, \beta) = (1, x, y, xy) = Q(\alpha_0, \beta_0), \tag{4}$$

where $yx=-xy$, $x^2=\alpha_0=\gamma$, $y^2=\beta_0=-\alpha\beta(\xi_2^2-\xi_3^2\alpha)$. By the Lemma above we must have $\delta=(\xi_4^2-\xi_5^2\gamma)\beta_0$ for ξ_4, ξ_5 in F. Combining this result with the Lemma we have the following result.

THEOREM. *A necessary condition that two normal simple algebras of degree two over any non-modular field* F,

$$A = Q(\alpha, \beta),\ B = Q(\gamma, \delta),$$

be equivalent is that there exist ξ_1, ξ_2, ξ_3 *in* F *for which*

$$\gamma = \xi_1^2\alpha + \xi_2^2\beta - \xi_3^2\alpha\beta \neq 0. \tag{5}$$

For any ξ_1, ξ_2, ξ_3 *satisfying* (5) *the algebras* A *and* B *are equivalent if and only if*

$$\delta = (\xi_4^2 - \xi_5^2\gamma)\beta_0, \tag{6}$$

with ξ_4 *and* ξ_5 *in* F, *where*

$$\beta_0 = \beta,\ \textit{or}\ \beta_0 = -\alpha\beta(\xi_2^2 - \xi_3^2\alpha), \tag{7}$$

according as ξ_2 *and* ξ_3 *are or are not both zero.*

We have therefore proved that all algebras $Q(\gamma, \delta)$ equivalent to a given A are obtained by (5), (6), (7) as $\xi_1, \cdots, \xi_5$ range over all quantities of F for which $\gamma\delta\neq 0$.

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* A rational proof holding for any F was given in my Bulletin paper (loc. cit.). This result was also used to prove the above lemma but by a rather complicated computation.