ON ϑ, ϕ IDENTITIES

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1. *Introduction.* In a recent series of papers* Bell has referred to several ϑ, ϕ identities which were derived at his suggestion by the writer. The purpose of this paper is to indicate the method of deriving these identities and to point out how they may be generalized.

2. *Formulas of Degree Three in Four Variables.* Consider the well known theta formula of Jacobi

$$(1)\qquad 2\vartheta_1(w)\vartheta_1(u)=\begin{vmatrix}\vartheta_0\left(\dfrac{w+u}{2},\ q^{1/2}\right), \vartheta_3\left(\dfrac{w+u}{2},\ q^{1/2}\right)\\ \vartheta_0\left(\dfrac{w-u}{2},\ q^{1/2}\right), \vartheta_3\left(\dfrac{w-u}{2},\ q^{1/2}\right)\end{vmatrix}.$$

From (1) we easily prove that $2\vartheta_1(w)\vartheta_1(u)\vartheta_3((u+z)/2, q^{1/2})$ equals

$$(2)\qquad \begin{vmatrix}\vartheta_0\left(\dfrac{w+u}{2},\ q^{1/2}\right)\vartheta_3\left(\dfrac{u+z}{2},\ q^{1/2}\right) \\ \quad -\vartheta_3\left(\dfrac{w+u}{2},\ q^{1/2}\right)\vartheta_0\left(\dfrac{u+z}{2},\ q^{1/2}\right), \vartheta_3\left(\dfrac{w+u}{2},\ q^{1/2}\right)\\ \vartheta_0\left(\dfrac{w-u}{2},\ q^{1/2}\right)\vartheta_3\left(\dfrac{u+z}{2},\ q^{1/2}\right) \\ \quad -\vartheta_3\left(\dfrac{w-u}{2},\ q^{1/2}\right)\vartheta_0\left(\dfrac{u+z}{2},\ q^{1/2}\right), \vartheta_3\left(\dfrac{w-u}{2},\ q^{1/2}\right)\end{vmatrix}$$

$$= 2\begin{vmatrix}\vartheta_1\left(\dfrac{w+z+2u}{2}\right)\vartheta_1\left(\dfrac{w-z}{2}\right), \vartheta_3\left(\dfrac{w+u}{2},\ q^{1/2}\right)\\ \vartheta_1\left(\dfrac{w+z}{2}\right)\vartheta_1\left(\dfrac{w-z-2u}{2}\right), \vartheta_3\left(\dfrac{w-u}{2},\ q^{1/2}\right)\end{vmatrix}.$$

* E. T. Bell, *Quadratic Partitions*, this Bulletin, Paper I, vol. 37 (1931), pp. 871–875; Papers II, III, and IV, vol. 38 (1932), pp. 551–554, pp. 569–572, and pp. 697–699. For notations undefined in the present paper, the reader is referred to Bell's four papers.

Equating the two values of $\vartheta_1(w)$ obtained from (2) and from the corresponding expression for $2\vartheta_1(w)\vartheta_1(v)\vartheta_3((v-z)/2),\ q^{1/2})$, dividing the resulting equation by $\vartheta_1((w+z)/2)\vartheta_1((w-z)/2)$, and simplifying, we establish the ϑ, ϕ identity

$$\begin{aligned}(3)\qquad &\phi_{111}\left(\frac{w+z}{2},\ u\right)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-u}{2},\ q^{1/2}\right)\\ &+\phi_{111}\left(\frac{w-z}{2},\ -u\right)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+u}{2},\ q^{1/2}\right)\\ &-\phi_{111}\left(\frac{w-z}{2},\ v\right)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-v}{2},\ q^{1/2}\right)\\ &-\phi_{111}\left(\frac{w+z}{2},\ -v\right)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+v}{2},\ q^{1/2}\right)\equiv 0.\end{aligned}$$

From (1) other identities obviously may be derived. In (2), for example, we may introduce the factor $\vartheta_0((u+z)/2),\ q^{1/2})$ instead of $\vartheta_3((u+z/2),\ q^{1/2})$, and use a theta formula similar to but different from (1) to complete the simplification; or we may combine the factors with the second column of the determinant in (1) instead of with the first. The resulting identities all have the same arguments u, v, z, $(w+z)/2$, $(v-z)/2$, etc., but the subscripts on the ϕ's and the ϑ's are changed.

Formula (1) is only one source of such identities, for a large number of Jacobi's formulas similar to (1) yield sets. Not all the identities, however, are independent. If in (3), for example, we replace q by $-q$ or in turn increase each of the variables by multiples of $\pi/2$ or π, we can write down a chain of new identities (involving the same arguments) until finally the original identity is restored. Let us denote (3) by

$$(4)\qquad (111,\ 33)+(111,\ 33)-(111,\ 33)-(111,\ 33)\equiv 0.$$

Replacing w by $w+\pi$ in (4), we get

$$(5)\qquad (221,\ 30)+(221,\ 30)-(221,\ 30)-(221,\ 30)\equiv 0,$$

and increasing each of w, z, u, v in (4) by $\pi/2$ gives us

$$(6)\qquad (122,\ 33)-(212,\ 30)+(212,\ 03)-(122,\ 00)\equiv 0.$$

3. *More General Cases.* Each of the identities (4), (5), and (6), exhibited above, is of degree three (one in ϕ and two in ϑ) and each contains four variables. In the following paragraphs we indicate how identities of greater degree or in more variables may be derived from any identity of type (4).

One method is to utilize well known relations such as the following:

$$
(7)\qquad
\begin{aligned}
\vartheta_0\vartheta_0(2a,\, q^2) &= \vartheta_0(a)\vartheta_3(a),\\
\vartheta_0\vartheta_1(2a,\, q^2) &= \vartheta_1(a)\vartheta_2(a),\\
\vartheta_2\vartheta_1(a,\, q^{1/2}) &= 2\vartheta_1(a)\vartheta_0(a),\\
\vartheta_2\vartheta_2(a,\, q^{1/2}) &= 2\vartheta_3(a)\vartheta_2(a),\\
\vartheta_2\vartheta_2(a,\, iq^{1/2}) &= 2e^{i\pi/4}\vartheta_2(a)\vartheta_0(a),\\
\vartheta_2\vartheta_1(a,\, iq^{1/2}) &= 2e^{i\pi/4}\vartheta_1(a)\vartheta_3(a).
\end{aligned}
$$

By means of (7), a ϕ may be expressed as the product of two ϕ's and a theta function as the product of two ϑ's. If ϕ_{111} in (4) be so expressed, we obtain

$$
(8)\qquad
\begin{aligned}
&\phi_{111}\left(\frac{w+z}{2},\, u\right)\phi_{222}\left(\frac{w+z}{2},\, u\right)\vartheta_3(v-z)\vartheta_3(w-u)\\
&+\phi_{111}\left(\frac{w-z}{2},\, -u\right)\phi_{222}\left(\frac{w-z}{2},\, -u\right)\vartheta_3(v-z)\vartheta_3(w+u)\\
&-\phi_{111}\left(\frac{w-z}{2},\, v\right)\phi_{222}\left(\frac{w-z}{2},\, v\right)\vartheta_3(u+z)\vartheta_3(w-v)\\
&-\phi_{111}\left(\frac{w+z}{2},\, -v\right)\phi_{222}\left(\frac{w+z}{2},\, -v\right)\vartheta_3(u+z)\vartheta_3(w+v)\equiv 0,
\end{aligned}
$$

an identity of degree four.

It is easily verified that

$$
(9)\qquad
\begin{aligned}
&\vartheta_1'^2\phi_{pqr}(a,\, b)\\
&\qquad = \phi_{pst}(a+x,\, b-x)\phi_{sqm}(a,\, x)\phi_{trn}(b,\, -x)\vartheta_m(x)\vartheta_n(-x),
\end{aligned}
$$

where each of the letters s, t, m, n is any one of the four theta subscripts.

This relation enables us not only to increase the degree, but also with each application, to increase the number of variables by one. Applying (9) to (4) we have the new identity

$$
\begin{aligned}
&\phi_{111}\left(\frac{w+z+2x}{2}, u-x\right)\phi_{111}\left(\frac{w+z}{2}, x\right)\\
&\quad\cdot\phi_{111}(u,-x)\vartheta_3\left(\frac{v-z}{2}, q^{1/2}\right)\vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right)\\
&+\phi_{111}\left(\frac{w-z+2x}{2}, -u-x\right)\phi_{111}\left(\frac{w-z}{2}, x\right)\\
&\quad\cdot\phi_{111}(-u,-x)\vartheta_3\left(\frac{v-z}{2}, q^{1/2}\right)\vartheta_3\left(\frac{w+u}{2}, q^{1/2}\right)\\
&-\phi_{111}\left(\frac{w-z+2x}{2}, v-x\right)\phi_{111}\left(\frac{w-z}{2}, x\right)\\
&\quad\cdot\phi_{111}(v,-x)\vartheta_3\left(\frac{u+z}{2}, q^{1/2}\right)\vartheta_3\left(\frac{w-v}{2}, q^{1/2}\right)\\
&-\phi_{111}\left(\frac{w+z+2x}{2}, -v-x\right)\phi_{111}\left(\frac{w+z}{2}, x\right)\\
&\quad\cdot\phi_{111}(-v,-x)\vartheta_3\left(\frac{u+z}{2}, q^{1/2}\right)\vartheta_3\left(\frac{w+v}{2}, q^{1/2}\right)\equiv 0.
\end{aligned}
$$

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