A. $(e''+e)\neq e$, by 8(ii). B. $(e''+e)\neq(a'+a)$, by 6(i). C. $(e''+e)\neq[b'+(a+b)]$. For otherwise, by 3, 9(i), 2 and 4, either (i) $e'=b$ and $e=(a+b)$, or else (ii) $e=b'$ and $e''=(a+b)$. But (i) is impossible since $(a+b)'\neq b$ by 5(ii), and (ii) is impossible since $e\neq b'$ by 8(i). D. $(e''+e)\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$. Indeed otherwise in view of 3, 11, 2 and 4, either (i) $e'=(b'+c)$ and $e=[(a+b)'+(a+c)]$ which contradicts 8(ii), or else (ii) $e''=[(a+b)'+(a+c)]$ and $e=(b'+c)'$ which contradicts 8(i) and also 11.

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# CONCURRENCE AND UNCOUNTABILITY*

BY N. E. RUTT

1. *Introduction.* The point set of chief interest in this paper, a plane bounded continuum $Z$, is the sum of a continuum $X$ and a class of connected sets $[X_\alpha]$, each element $X_a$ of which has at least one limit point in $X$ and is a closed subset of $c_u(X+X_b)$, where $X_b$ is any element of $[X_\alpha]$ different from $X_a$ and where $c_u(X+X_b)$ is the unbounded component of the plane complement of the set $X+X_b$. Upon a basis of separation properties, order† may be assigned to the elements of $[X_\alpha]$ agreeing in its details with that of some subset of a simple closed curve. We shall use some definite element $X_r$ of $[X_\alpha]$ as reference element, selecting as $X_r$ one of $[X_\alpha]$ containing a point arcwise accessible from $c_u(Z)$. A countable subcollection $[X_i^h]$ of $[X_\alpha]$ excluding $X_r$ is called a *series* if for each $j$, $(j=2, 3, 4, \cdots)$, the elements $X_j$ and $X_r$ separate $X_{j-1}$ and $X_{j+1}$. Two different series $[X_i^h]$ and $[X_i^k]$ are said to be *opposite in sense* if there exist different subscripts $m$ and $n$ such that $X_m^h$ and $X_m^k$ separate both $X_n^h$ and $X_n^k$ from $X_r$; otherwise they are said to have the *same sense*. They are said to be *concurrent* if they have the same sense and if there exists no element of $[X_\alpha]$ which together

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† R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202; J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562; N. E. Rutt, *On certain types of plane continua*, Transactions of this Society, vol. 33, No. 3, pp. 806–816.

with $X_r$ separates infinitely many of the one from infinitely many of the other. It is easily seen that two series with infinitely many elements in common are concurrent, that two non-concurrent series may exist such that no element of $[X_\alpha]$ separates infinitely many of one from infinitely many of the other with respect to $X_r$, and that when two series having the same sense are not concurrent, then one of the two contains an element which together with $X_r$ separates all the elements of one series from all but a finite number of the elements of the other. This paper deals mainly with collections $[[X_i]_\alpha]$ of such series as $[X_i^h]$. Sets whose elements are series of this sort have some properties which are close analogs of properties of the collection $[X_\alpha]$. For instance, when no two of four given elements of $[[X_i]_\alpha]$ are concurrent, then some pair of the four will separate the other pair in a sense easily distinguished.

THEOREM 1. *If* $[[X_i]_\alpha]$ *is a collection of series of* $[X_\alpha]$, *no two of which are opposite in sense and no two of which are concurrent, then* $[[X_i]_\alpha]$ *is not both well-ordered and uncountable.*

If we suppose otherwise, we arrive at a contradiction. Consider the element $[X_i]_\lambda$ of the well-ordered collection $[[X_i]_\alpha]$. If $\lambda$ is any transfinite ordinal of the type $\mu+n$, where $n$ is a positive integer and $\mu$ is a transfinite ordinal not of this type, then $[X_i]_\lambda$ contains an element $X_{i_\lambda}$ which, together with $X_r$, separates all the elements of every series $[X_i]_\rho$ of $[[X_i]_\alpha]$ having $\rho$ preceding $\lambda$ from every element $X_j$ of $[X_i]_\lambda$ having $j>i_\lambda$. Let $[Y_i]_\lambda$ be that series of $[X_\alpha]$ whose elements are the elements of $[X_i]_\lambda$ that have subscripts greater than $i_\lambda$. Let $[[Y_i]_\beta]$ be the collection of all series $[Y_i]_\beta$ which may be obtained from such elements of $[[X_i]_\alpha]$ as $[X_i]_\lambda$ with $\lambda$ as specified above. The collection $[[Y_i]_\beta]$ is well-ordered, and consists of elements no two of which are concurrent or opposite in sense. That it is also uncountable may be seen directly from the fact that following immediately every element in $[[X_i]_\alpha]$ which does not contribute an element to $[[Y_i]_\beta]$ is one which does. Accordingly $[X_\beta]$, that maximal subcollection of $[X_\alpha]$ each element of which is included in an element of $[[X_i]_\beta]$, is likewise well-ordered and uncountable. Now $[X_\beta]$ includes an uncountable subcollection $[Y_\beta]$, likewise well-ordered, each element of which contains a point at a distance from $X$ greater than some definite positive quantity $\epsilon$. Let $C$ be a simple closed curve enclosing $X$ whose interior contains no point at a distance greater than $\epsilon/2$ from some point of $X$. Each element of $[Y_\beta]$ has points within and without $C$. For each element $Y_b$ of $[Y_\beta]$ let $y_b$ be a point of $C$ which is limit of that component of the subset of $X+Y_b$ interior to $C$ which contains $X$. The collection $[y_b]$ is both well-ordered* and uncountable, which constitutes a contradiction.†

2. THEOREM 2. *If* $[[X_i]_\alpha]$ *is a collection of series, no two of which are concurrent or opposite in sense, and for every element* $[X_i]_a$ *of* $[[X_i]_\alpha]$ *there is a point a of the element* $X_a$ *of* $[X_\alpha]$ *different from* $X_r$ *within every neighborhood of which there is a point of some element of* $[X_i]_a$, *then* $[[X_i]_\alpha]$ *is well-ordered.*

It will be shown that, given any element $[X_i]_a$ of $[[X_i]_\alpha]$, there is a first element of $[[X_i]_\alpha]$, infinitely many of whose elements separate all of $[X_i]_a$ from $X_a$ with respect to $X_r$. If we suppose that there is none, a contradiction will be obtained. This supposition implies that $[[X_i]_\alpha]$ includes a countable sequence $[[X_i]_j]$ such that, owing to the fact that in every one of $[[X_i]_\alpha]$, when $m>n$, then the $m$th element separates the $n$th from $X_a$ with respect to $X_r$‡, for $j=1, 2, 3, \cdots$, all but at most a finite number of $[X_i]_{j+1}$ separate all of $[X_i]_a$ from $X_a$ and from all but a finite number of $[X_i]_j$. It would thus be allowable to suppose that all of $[X_i]_{j+1}$ separate all of $[X_i]_a$ from all of $[X_i]_j$; and, for simplification, this will be assumed. Consider the prime ends,§ a simple closed curve $C$ of $c_u(X_r+X+X_a)$. The subcollection of these, each one of which contains among its chief points a point of $X_a$, is an arc‖ $C_a$, not including its ends. Let the ends of $C_a$ be $U$ and $V$, let $R$ be any element of $C$ with a chief point in $X_r$, and let $C_u$ and $C_v$ be the

* *On certain types of plane continua*, p. 809, loc. cit.

† C. Zarankiewicz, *Ueber die Zerschneidungspunkte der zusammenhängender Mengen*, Fundamenta Mathematicae, vol. 12, p. 121, Hilfsatz.

‡ *On certain types of plane continua*, Corollary 2, loc. cit.

§ Defined by C. Carathéodory in his paper, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

‖ N. E. Rutt, *Prime ends and order*, Part 1, §10. This paper has been accepted for publication by the Annals of Mathematics, but is not yet in print.

subarcs of $C$ complementary to $R$ and $C_a$ including their ends, $U$ being contained by $C_u$ and $V$ by $C_v$. For each value of $j$, $(j=1, 2, 3, \cdots)$, let $S_j$ be the set intercepted* by $X_a$ and $[X_i]_j$, let $M_j$ be the subcollection of $U+C_a+V$ each element of which is limit of $S_j$, but not of $\sum_i X_i^j$, let $N_j$ be the subcollection of $U+C_a+V$ each element of which is limit of $\sum_i X_i^j$, and let $V_j$ be the sum of $N_j$ and the complement in $U+C_a+V$ of $M_j$. If it be supposed that $C_u$ includes every element of $C$ which is limit of any one of $[X_\alpha]$ contained in a series of $[X_i]_j$,† then $V_j \supset V$. Moreover, as $S_1 \subset S_2 \subset \cdots \subset S_i \subset S_{i+1} \subset \cdots$, then $M_1 \subset M_2 \subset \cdots \subset M_1 \subset M_{i-1} \subset \cdots$, and the elements of $[V_j]$ are arcs with a set $H_a$ in common, such that $V_1 \supset V_2 \supset \cdots \supset V_i \supset V_{i+1} \supset \cdots \supset H_a$. If $[N_j]$ is a collection no infinite subset of which has any common element, then there must be a prime end $H$ in $H_a$ which, considered as a collection of domains, includes no element $\eta$ which does not contain a prime end belonging to some one of $[N_j]$. If $H$ contains a point $h$ of $X_a$, then $h$ is a point which, although not limit of any one of the point sets $[S_j]$ through $H$, is nevertheless limit of the collection $[S_j]$ through $H$. Let $[\tau_i]$ be a monotonic collection of neighborhoods of $h$ whose only common point is $h$, so chosen with respect to an arbitrary element $\eta$ of $H$ that there exists an infinite subset $[T_i]$ of $[S_i]$ having, for each $i$, $\eta\cdot\tau_i\cdot T_i \neq 0$ and $\eta\cdot\tau_{i+1}\cdot T_i=0$. Under these circumstances, $[X_\alpha]$ contains a set $[Y_i]$, where $Y_i\cdot\tau_i\neq 0$ and $T_i \supset Y_i$, so that $h$ is a limit point of the point set $\sum Y_i$. But $[Y_i]$ is a series, because for each value of $i$, $(i=1, 2, 3, \cdots)$, $T_{i+1}$ contains $Y_{i+1}$, whereas $T_i$ does not; and all elements of $T_{i+1}$ not belonging to $T_i$ are separated from $X_a$ by $X_r$ and any element whatever of $T_i$. For the same reason, $[Y_i]$ is a series in which $Y_i$ and $X_r$ separate $Y_{i+1}$ and $X_a$. As the point $h$ of $X_a$ is limit of $\sum Y_i$, this is a contradiction,‡ which proves the theorem in this case.

If $H$ contains no point of $X_a$, then it is $V$, and although belonging to $C_v$, is limit of a series like $[Y_i]$ selected from $[X_\alpha]$, not by means of a collection $[\tau_i]$ of neighborhoods of a point, but by means of a chain of domains defining $H$; and this is also

* For definitions and properties used here see *Prime ends and order*, Part 3, loc. cit.

† *Prime ends and order*, Part 2, §6, loc. cit.

‡ *On certain types of plane continua*, Corollary 2, loc. cit.

contradictory.* On the other hand, in case infinitely many of $[N_j]$ contain the element $H$ of $C_a$, then $H$ contains a point $h$ of $X_a$ which is shown, much as above, to be limit of a series $[Y_i]$, where $Y_i$ is an element of $T_i$ but not of $T_{i-1}$; this constitutes a contradiction, as above.

COROLLARY 1. *If $[[X_i]_\alpha]$ is a collection of series no two of which are concurrent, and for every element $[X_i]_a$ of $[[X_i]_\alpha]$ there is a point a of the element $X_a$ of $[X_\alpha]$ different from $X_r$ within every neighborhood of which there is a point of some element of $[X_i]_a$, then $[[X_i]_\alpha]$ is countable.*

If there are elements of $[[X_i]_\alpha]$ opposite in sense, then $[[X_i]_\alpha]$ consists of two subcollections having the property that no pair of elements belonging to the same one can be either concurrent or opposite in sense. Thus, whether or not there are in $[[X_i]_\alpha]$ two elements opposite in sense, the corollary follows easily from Theorem 1 because when there are two such elements each of the two collections mentioned is countable.

3. *An Application.* We shall now give an application of the foregoing results.

THEOREM 3. *If $X_a$ is any element of $[X_\alpha]$, then $Z-X_a$ is the sum of a countable set of continua, each one of which is of type $Z$.*

Suppose at the outset that $X_a$ contains a point arcwise accessible from the unbounded complementary domain of $Z$. If $Z-X_a$ is closed, the theorem is true; whereas, if it is not, then there must be a series of elements $[X_i]_1$ of $[X_\alpha]$ each one arcwise accessible from $c_u(Z)$ among whose limit points is a point of $X_a$. Let $[X_\alpha]_1$ be the subset of $[X_\alpha]$ consisting of all of its elements which are separated from $X_a$ by some pair of the elements $X_r, X_1^1, X_2^1, \cdots, X_i^1, \cdots$. The set $Z-\Sigma X_\alpha^1$ is clearly a continuum $Z_1$ containing $X_a$. If $Z_1-X_a$ is not closed, $Z_1$ contains a subseries $[X_i]_2$ of elements of $[X_\alpha]$, each accessible from $c_u(Z_1)$ with a limit point in $X_a$; for, if there were no such series, no point of $X_a$ could be limit of $c_u(Z_1)$, and thus no point of $X_a$ could be arcwise accessible from $c_u(Z)$. There is thus a set $[X_\alpha]_2$ and a set $Z_2=Z_1-\Sigma X_\alpha^2$ which is closed and contains $X_a$. In fact, there are three series $[X_i]_1, [X_i]_2, [X_i]_3, \cdots; [X_\alpha]_1, [X_\alpha]_2,$

* *On certain types of plane continua*, Corollary 3, loc. cit.

$[X_\alpha]_3, \cdots$; and $Z_1, Z_2, Z_3, \cdots$. Let $W_\omega = \Pi Z_i$. Clearly $W_\omega$ is a continuum of type $Z$ containing $X+X_a$. If $W_\omega - X_a$ is not closed, $W_\omega$ contains a series $[X_i]_\omega$ of elements of $[X_\alpha]$, each arcwise accessible from $c_u(W_\omega)$, having a limit point in $X_a$, so that there is a collection $[X_i]_\omega$ and a continuum $Z_\omega$ resembling $Z_1$. In short, the process described may be continued until a set $Z_\sigma$ is obtained having $Z_\sigma - X_a$ closed. If $Z_\sigma$ is not a set that has been obtained in the way that $W_\omega$ was obtained, then the order type of $Z_1, Z_2, \cdots, Z_\omega, \cdots, Z_\sigma$ is the same as that of $[X_i]_1, [X_i]_2, \cdots, [X_i]_\omega, \cdots, [X_i]_\sigma$; whereas, if not, then the order type of the collection $[Z_\lambda]$ may be obtained from that of $[[X_i]_\lambda]$ by the addition of the single transfinite ordinal $\sigma$. However, in either case, $[Z_\lambda]$ is a countable collection because, owing to Corollary 1, $[[X_i]_\alpha]$ is countable.

The set $Z-X_a$ may now be expressed as the sum of a countable collection of sets as follows. Let the set $K_0$ be $X+X_r+X_1^1$ plus all the elements of $[X_\alpha]$ which are separated from $X_a$ by $X_1^1$ and $X_r$. Let $K_1$ be $X+X_1^1+X_2^1$ plus all elements of $[X_\alpha]$ separated from both $X_a$ and $X_r$ by $X_1^1$ and $X_2^1$. Let $K_n$, $(n=2, 3, 4, \cdots)$, be $X+X_n^1+X_{n+1}^1$ plus all elements of $[X_\alpha]$ separated from $X_a$ by $X_n^1$ and $X_{n+1}^1$. In general, as to $K_\lambda$, if $\lambda$ is a transfinite ordinal of the form $\mu+n$, where $n$ is a finite positive integer and $\mu$ is a transfinite ordinal not of the same form as $\lambda$, then let $K_\lambda$ be $X+X_n^\mu+X_{n+1}^\mu$ plus all elements of $[X_\alpha]$ separated from $X_a$ by $X_n^\mu$ and $X_{n+1}^\mu$; while, if $\lambda$ is not of this form, then let $K_\lambda$ consist of $X$ and all of $[X_\tau]$, where $X_t$, any element of $[X_\tau]$, is $X_1^\lambda$ or any element of $[X_\alpha]$ separated from $X_a$ by $X_1^\lambda$, and $[X_i]_\beta$, where $[X_i]_\beta$ is any element of $[[X_i]_\lambda]$ not opposite in sense to $[X_i]_\lambda$ with $\beta$ a transfinite ordinal preceding $\lambda$. The collection $[K_\lambda]$ is clearly countable, since it has the same cardinal number as the collection $[X_{i\lambda}^\lambda]$ of all the elements of $[X_\alpha]$ included in one of $[[X_i]_\lambda]$.

Consider the set $K_\lambda$. It is obviously connected. If it is not closed, let $l$ be a limit point of it. Now if $\lambda=\mu+n$, $n$ and $\mu$ being as in the paragraph above except that $\mu$ may possibly be zero, then $K_\lambda \subset Z_\mu$, so that the point $l$ must belong to some element $X_l$ of $[X_\alpha]$ contained in $Z_\mu$. All but two of the elements of $[X_\alpha]$ in $Z_\mu$ belonging to $K_\lambda$ are separated from $X_a$ by $X_{n+1}$ and $X_n$, both of these being elements of $[X_\alpha]$ in $Z_\mu$ arcwise accessible from $c_u(Z_\mu)$, so, as $X_l$ contains $l$, it can not be separated from them by $X_n$ and $X_{n+1}$. Thus $l$ may exist only if $n=0$. But in this case $X_l$ would have to be separated from any series of $[X_\alpha]$ in $K_\lambda$ of which it contains a limit by $X_1{}^\lambda$ and $X_r$, both of these being elements of $[X_\alpha]$ in $K_\lambda$ arcwise accessible from $c_u(Z_\lambda)$. So $l$ can not exist, and $K_\lambda$ is closed. The statements above apply directly to all except $K_0$, which is easily seen to be a continuum by similar means. Accordingly, when $X_a$ contains a point arcwise accessible from $c_u(Z)$, the fact that $Z-X_a=\Sigma K_\lambda+(Z_\sigma-X_a)$ verifies the theorem.

In case $X_a$ contains no point arcwise accessible from $c(Z)$, let $[X_i]_1$ be a series of $[X_\alpha]$, such that, for $i=1, 2, 3, \cdots, X_r$ and $X_{i+1}^1$ separate $X_a$ and $X_i^1$, each element of $[X_i]_1$ is arcwise accessible from $c(Z)$, and there is none of $[X_\alpha]$ arcwise accessible from $c_u(Z)$ together with $X_r$ separating $X_a$ from more than a finite number of $[X_i]_1$. Suppose that, in addition to satisfying requirements specified earlier, $X_r$ has also been selected so as to be separated from $X_a$ by the elements $X_b$ and $X_c$ likewise arcwise accessible from $c_u(Z)$. If we omit all those of $[X_\alpha]$ separated from both $X_a$ and $X_r$ by either $X_b$ or $X_c$ and some element of $[X_i]_1$, a subcontinuum $\overline{Z}_1$ containing $X_a$ results. If $\overline{Z}_1-X_a$ is not closed and $X_a$ contains no point arcwise accessible from $c_u(\overline{Z}_1)$, the step above may be repeated, and, under similar circumstances, may be repeated indefinitely, with an occasional inserted step of finding $\Pi\overline{Z}_i$, until eventually there results a continuum $\overline{Z}_\sigma$ in which either $\overline{Z}_\sigma-X_a$ is closed or $X_a$ contains a point arcwise accessible from $c_u(\overline{Z}_\sigma)$. The collection $[[X_i]_\lambda]$ of series used in determining $\overline{Z}_\sigma$ consists of two subaggregates in each of which no two elements can be concurrent or opposite in sense; hence, from Theorem 1, it follows easily that $[[X_i]_\lambda]$ is countable. Consequently, very much as above, in the case already discussed, it may be seen that $(Z-\overline{Z}_\sigma)+X$ is the sum of a countable set of continua of type $Z$, so that since $\overline{Z}_\sigma-X_a$ has already been seen to be the sum of a countable set of continua of type $Z$, then $Z-X_a$ is also.

COROLLARY 2. *If $[X_n]$ is a finite subset of $[X_\alpha]$, then $Z-\Sigma X_n$ is the sum of a countable set of continua each of type $Z$.*

For $Z-X_1$ is the sum of a countable collection $[K_\alpha]_1$ of the sort required, so consider the distribution of the remaining members of $[X_n]$ among those of $[K_\lambda]_1$. If no more than one of $[X_n]$

is contained in any one of $[K_\lambda]_1$, the corollary follows easily from Theorem 3. If, on the other hand, $K_p^1$ of $[K_\lambda]_1$ were to contain more than one of $[X_n]$, $X_q$ being that one of $[X_n]$ lowest in subscript which it contains, then $K_p^1 - X_q$ would be the sum of a countable set of continua of type $Z$, so that, after taking due account of the fact that $X_q$ might belong to two different elements of $[K_\lambda]_1$ (but not to more than two), it would appear that $Z-(X_1+X_q)$ was also sum of a countable set $[K_\lambda]_2$ of continua of type $Z$. The process can be carried through a finite number of steps to prove the corollary.

Corollary 3. *If $[X_\lambda]$ is a countable subset of $[X_\alpha]$ not including $X_r$, and $[X_\alpha]$ contains a countable collection of pairs of elements, such that each element of every pair contains a point arcwise accessible from $c_u(Z)$, no pair separates from $X_r$ either any element contained in any other pair or more than a finite number of $[X_\lambda]$, and no element of $[X_\lambda]$ is not separated by some pair from $X_r$, then $Z-\Sigma X_\lambda$ is the sum of a countable collection of continua each of type $Z$.*

This corollary follows directly from Corollary 2 if we express $Z$ as the sum of a countable collection of continua each of which, except the one containing $X_r$, consists of $X$, a pair, and all of $[X_\alpha]$ separated from $X_r$ by the pair.

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