NOTE ON SETS OF POSITIVE MEASURE*

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A recurring question concerning (L-measurable) sets of positive measure is what properties they have in common with the linear interval. The following theorem is concerned with such a property, stated for sets of n-dimensional positive measure lying in euclidean n space.

THEOREM. Let \( A_1, A_2, \ldots, A_p \) be \( p \) sets of positive measure lying in euclidean n space. Then there exist \( p \) n-dimensional spheres \( S_1, S_2, \ldots, S_p \) such that for every set of \( p \) points \( s_\nu \), \((\nu = 1, 2, \ldots, p)\), belonging respectively to these spheres, there exists a set of \( p \) points \( a_\nu \), \((\nu = 1, 2, \ldots, p)\), lying respectively in \( A_1, A_2, \ldots, A_p \), such that the sets \( \{a_\nu\} \) and \( \{s_\nu\} \) are congruent. Moreover, there exists a set of \( p \) congruent spheres \( S_\nu \) satisfying the condition just stated and a positive number \( \delta \) such that for every selected \( \{s_\nu\} \), with \( s_\nu \) belonging to \( S_\nu \), the associated \( \{a_\nu\} \) may be so chosen that \( a_1 \) ranges over a set of measure \( >\delta \).

PROOF. Since \( A_1 \) is of positive measure, there is a sphere \( S'_1 \) in which the relative measure of \( A_1 \) is greater than \( 1 - \epsilon \), where \( \epsilon \) is a given positive number less than 1; that is, \( m(A_1, S'_1)/m(S'_1) > 1 - \epsilon, m(A) \) standing for the measure of \( A \). We may suppose, and we do so for simplicity of statement, that all the \( S'_\nu \), \((\nu = 1, \ldots, p)\), are equal, and we denote their common measure by \( \mu \), and their respective centers by \( c_\nu \). Let \( p \) be a positive number such that if a sphere of measure \( \mu \) is translated a distance \(<\rho \), the part belonging to the sphere in both positions is of measure \( (1 - \epsilon)\mu \). Denote by \( v_\nu \), \((\nu = 1, \ldots, p-1)\), the vector represented by the segment \( c_\nu c_{\nu+1} \); and let \( w_\nu \), \((\nu = 1, \ldots, p)\), be a given set of \( n \)-dimensional vectors, each of length \(<\rho \). If a set \( A \) (or point \( a \)) is given a displacement represented by the vector \( \pm v \), we denote the set (or point) in its new position by \( A \pm (v) \) (or \( a \pm (v) \)). Writing \( A_\nu S'_\nu = T''(v) \) and \( T' = T'_1 \), we set

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T'_1 + (v_1 - w_1 + w_2) = T'_2; T'_2 T'''' = T'''_{1''};
\]

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For the measures of the $T_\lambda^{(p)}$'s, we have the following inequalities: $m(T_1') > (1 - \epsilon)\mu$; $m(T_1''') > (1 - 4\epsilon)\mu$, since $m(T_1''') > (1 - \epsilon)\mu$ and the lengths of $w_1$ and $w_2$ are less than $\rho$; $m(T_1'''') > (1 - 7\epsilon)\mu$, and so on. We may thus conclude that $m(T_1^{(p)}) > [1 - (3\rho - 2)\epsilon]\mu$, which is positive if $\epsilon$ is taken small enough. We now define the spheres $S_1, \ldots, S_p$ of our theorem as of radii all less than $\rho$, and such that their respective centers $\gamma_\nu$ satisfy the relations: vector $\gamma_{\nu}\gamma_{\nu+1} = v_\nu, (\nu = 1, \ldots, p - 1)$. If now $s_\nu, (\nu = 1, \ldots, p)$, is a point chosen from $S_\nu$, we let vector $\gamma_\nu s_\nu = w_\nu$. Let $a_p$ be a point of $T_1^{(p)}$, which, as we have seen, is not empty if $\epsilon$ is sufficiently small. We then define $a_{p-1}, a_{p-2}, \ldots, a_1$ by the relation

$$a_\nu = a_{\nu+1} + (v_{\nu-1} - w_{\nu-1} + w_\nu), \quad (\nu = 2, 3, \ldots, p).$$

Since $a_p$ belongs to $T_1^{(p)}$, it belongs to $A_p$ and also to $T_2^{(p-1)}$; hence $a_{p-1}$ belongs to $T_1^{(p-1)}$ and therefore to $A_{p-1}$ and $T_2^{(p-2)}$; hence $a_{p-2}$ belongs to $T_1^{(p-2)}$, and so on. We conclude that $a_\nu, (\nu = 1, \ldots, p)$, belongs to $A_\nu$. Since $s_\nu$ satisfies the equation $s_\nu = s_{\nu+1} + (v_{\nu-1} - w_{\nu-1} + w_\nu), (\nu = 2, \ldots, p)$, we see that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Furthermore, since $a_p$ is an arbitrary point of $T_1^{(p)}$, whose measure is arbitrarily near $\mu$, we can satisfy the last condition of our theorem by taking, for example, $\delta = \mu/2$, if $\epsilon$ is small enough.

If, in particular, the $p$ given sets $A_\nu$ are identical, we may take the spheres $S_\nu'$ as identical, thus reducing the vectors $v_\nu$ to zero. The spheres $S_\nu$ may therefore be taken as identical, and we have the following corollary.

**Corollary.** If $A$ is an $n$-dimensional set of positive measure, there exists an $n$-dimensional sphere $S$ such that for every finite subset of $S$ there is a congruent subset of $A$.

If $A$ is a one-dimensional set, we obtain the theorem of Steinhaus,* that the set of distances between pairs of points of a

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* Sur les distances des points, Fundamenta Mathematicae, vol. 1 (1920), p. 99. A simpler proof of this theorem, close in idea to our own, was published by Ruziewicz (after the present paper was read), Fundamenta Mathematicae, vol. 7 (1925), p. 141.
(linear) set of positive measure contains an interval with 0 as left end point.

We have proved that if \( S_1 \) is a set of positive measure and \( S_2 \) a finite set, there is a subset of \( S_1 \) similar* to \( S_2 \). To what extent can the condition of finiteness of \( S_2 \) be modified if the theorem is to remain valid? Since every set of positive measure contains a perfect subset of positive measure, it follows that if every set of positive measure contains a set similar to \( S_2 \), it contains a set similar to \( S_2 + S_2' \), where \( S_2' \) is the derivative of \( S_2 \). It thus suffices to restrict \( S_2 \) to being closed. Or we may restrict \( S_2 \) to being denumerable, since every set contains a denumerable subset which is dense in it. Not all sets of positive measure can contain a set similar to \( S_2 \) if \( S_2 \) is not non-dense. Since we naturally restrict \( S_2 \) to being bounded, we are led to ask: What bounded, non-dense sets \( S_2 \) are such that every set of positive measure contains a set similar to \( S_2 \)? That this property is not shared by every bounded, non-dense \( S_2 \), and therefore not by every bounded, non-dense, denumerable set, is shown by the following fact.

**Theorem.** If \( S_1 \) is a given bounded, non-dense, perfect set, there exists a perfect set \( S_2 \) of zero measure such that no subset of \( S_1 \) is similar to \( S_2 \).

While this theorem is meant to refer to \( n \)-dimensional sets, we assume in the proof that \( S_1 \) and \( S_2 \) are linear sets, there being no essential difference in the argument for \( n \)-dimensional sets. We suppose, as we may, that the given set \( S_1 \) lies in the interval \((0, 1) = I\). Let \( C(S_1) \) be the complement of \( S_1 \) in \( I \); \( \Delta \) a variable subinterval of \( I \); \( \mu_1(\Delta) \) the ratio of the maximum length of a connected portion of \( C(S_1) \) in \( \Delta \) to the length of \( \Delta \); and \( \sigma_1(h) \), for \( h \) a given positive number, the greatest lower bound of \( \mu_1(\Delta) \) for all subintervals \( \Delta \) of \( I \) of length \( h \). Then \( \sigma_1(h) \) is a positive, continuous function of \( h \). The perfect set \( S_2 \) will be defined as the complement in \( I \) of the set of intervals \( \Delta_m,i \), which are defined as follows: Insert in \( I \) a set of equally spaced intervals \( \Delta_{i,i}, (i = 1, 2, \cdots, m_1) \), of equal length \( l_1 \), such that \( m_1 l_1 = 1/2 \), the equality of spacing being understood in the sense that the

* We are using "similar" in the ordinary euclidean sense of the existence of a biunique correspondence with invariant ratio of distances.
space between any two adjacent \( \Delta_{1i} \) shall be equal to the spaces between 0, 1 and the first, last \( \Delta_{1i} \), respectively; moreover, \( m_1 \) is to be so large that \( \sigma_2(\lambda_1) < \sigma_1(\lambda_1) \), where \( \lambda_1 = 1 \), and \( \sigma_2 \) has the same meaning for the set \( S_2 \), now being defined, as \( \sigma_1 \) for \( S_1 \). Similarly insert in each of the intervals \( \Delta_{1i}^i, (i = 1, 2, \cdots, m_1 + 1) \), of length \( l_1' \), that are complementary to the \( \Delta_{1i} \), the same number of equally-spaced intervals \( \Delta_{2i}, (i = 1, 2, \cdots, m_2) \), of equal length \( l_2 \), where \( m_2 \) signifies the total number of the \( \Delta_{2i} \) in all the \( \Delta_{1i} \); moreover, the \( \Delta_{2i} \) are to be such that \( m_2 l_2 = 1/4 \), and \( m_2 \) so large that \( \sigma_2(\lambda_2) < \sigma_1(\lambda_2) \) for \( 1 \leq t \leq 2 \), where \( 2\lambda_2 = l_1' \). In general, let \( \{\Delta_{n-1,i}^n\} \) be the set of intervals of length \( l_{n-1}' \), complementary to the set of all \( \Delta_{n,i} \), \( n \leq n - 1 \). Insert in each \( \Delta_{n-1,i} \) the same number of equally spaced intervals \( \Delta_{ni} \) of equal length \( l_n \), such that \( m_n l_n = 1/2^n \), \( m_n \) being the total number of intervals \( \Delta_{ni} \) and \( m_n \) so large that \( \sigma_2(\lambda_n) < \sigma_1(\lambda_n) \) for \( 1 \leq t \leq n \), where \( n\lambda_n = l_n - l_n' \). Suppose now that \( S_3 \) is any set whatsoever lying in \( I \) and similar to \( S_2 \); then \( S_3 \) cannot lie in \( S_1 \). For let \( k \) be the ratio of corresponding lengths in \( S_2 \) and \( S_3 \), and \( n \) an integer greater than \( k \). If \( \epsilon \) is a given positive number, we can find an interval \( \Delta \) of length \( \lambda_n \) lying between two points of \( S_3 \), and such that \( |\mu_3(\Delta) - \sigma_3(k\lambda_n)| < \epsilon \), \( \mu_3 \) having the same meaning for \( S_3 \) as \( \mu_1 \) for \( S_1 \). Hence, on account of the inequality \( \sigma_3(k\lambda_n) < \sigma_1(\lambda_n) \), we conclude that \( \mu_3(\Delta) < \sigma_1(\lambda_n) \leq \mu_1(\Delta) \), if \( \epsilon \) is small enough. That is to say, the maximum length of a connected portion of \( C(S_3)\Delta \) is less than such maximum length for \( C(S_1)\Delta \), and therefore \( S_3 \) cannot lie in \( S_1 \).

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