A NOTE ON THE DICKSON THEOREM ON UNIVERSAL TERNARIES*

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1. *Introduction*. A form f with integer coefficients in integer variables is called *universal* if it represents *all* positive and negative integers. Evidently, since f is homogeneous, it represents zero for the variables all zero. In case $f=0$ for integral values of the variables not all zero f is called a zero form.

L. E. Dickson† has given a number-theoretic proof of his theorem that *every universal ternary quadratic form is a zero form*. But his proof is highly technical and consequently quite long and complicated. In the present note I shall give an almost trivial rational proof of Dickson's result. I shall also prove a generalization of his theorem for ternaries over any non-modular field F.

2. *Quadratic Forms over F*. Let F be any non-modular field and let $f(x_1, \cdots, x_n)$ be an n-ary quadratic form over F. Then we shall call f a *zero form* if $f=0$ for $x_1, \cdots, x_n$ in F and not all zero. We shall also say that, if every ρ in F is represented by f for $x_1, \cdots, x_n$ in F, the form f is universal over F.

It is well known‡ that there exists a non-singular linear transformation $x_i = \sum a_{ij} X_j$ with a_{ij} in F such that

$$f(x_1, \cdots, x_n) \equiv \phi(X_1, X_2, \cdots, X_n) \equiv \sum_{i=1}^{r} g_i X_i^2 + 0 \cdot \sum_{j=r+1}^{n} X_j^2,$$

with $g_i \neq 0$ in F. The integer r is the rank of f. Evidently f is a zero form if and only if ϕ is a zero form. But if $r<n$, the form ϕ vanishes for any X_n in F, if $X_1 = \cdots = X_r = 0$.

THEOREM 1. *Every n-ary of rank $r<n$ is a zero form. Every n-ary of rank n is equivalent to*

$$g_1X_1^2 + g_2X_2^2 + \cdots + g_nX_n^2, \qquad (g_i \text{ in } F),$$

with g_i all not zero.

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* Presented to the Society, April 15, 1933.

† See his *Studies in the Theory of Numbers*, pp. 17–21.

‡ See Dickson, *Modern Algebraic Theories*, p. 69

3. *Proof of the Dickson Theorem.* Let $f(x, y, z)$ be a universal ternary. By Theorem 1 either f is a zero form of rank less than three or

$$(1) \qquad f(x, y, z) \equiv \phi(X, Y, Z) \equiv \alpha X^2 + \beta Y^2 - \gamma Z^2,$$

where α, β, γ are rational, $\alpha\beta\gamma \neq 0$, and X, Y, Z are linearly independent rational linear functions of x, y, z. Define

$$(2) \quad \delta \equiv \gamma(\alpha\beta)^{-1}, \quad a \equiv \alpha\delta, \quad b \equiv \beta\delta, \quad -ab = -(\alpha\beta\delta)\delta = -\gamma\delta,$$

so that, for a rational number $\delta \neq 0$,

$$(3) \qquad \delta f \equiv \delta\phi \equiv \psi(X, Y, Z) \equiv aX^2 + bY^2 - abZ^2.$$

Write $\delta = \delta_1\delta_2^{-1}$, where δ_1 and δ_2 are integers. Since f is universal, $f(x, y, z) = \delta_1\delta_2$ for integer x, y, z. Then if $x_0 = x\delta_1^{-1}$, $y_0 = y\delta_1^{-1}$, $z_0 = z\delta_1^{-1}$, we have $f(x_0, y_0, z_0) = \delta_1^{-2}\delta_1\delta_2 = \delta_2\delta_1^{-1} = \delta^{-1}$, for rational x_0, y_0, z_0. Hence we have proved the following fact.

LEMMA 1. *If f is universal, $\phi = \delta^{-1}$ for rational X, Y, Z.*

Let then $\delta^{-1} = \phi$, $\psi = \delta\phi = \delta\delta^{-1} = 1 = aX^2 + bY^2 - abZ^2$, and write as a consequence

$$(4) \qquad \xi \equiv 1 - aX^2 = bY^2 - abZ^2.$$

If $\xi = 0$, put $\eta = 1$, $\zeta = X$, so that

$$\psi(\xi, \eta, \zeta) = b \cdot 1^2 - ab \cdot X^2 = b(1 - aX^2) = b\xi = 0$$

for $\eta \neq 0$, and $\phi = \delta^{-1}\psi$ is a zero form. Hence f is a zero form, since $f = 0$ for rational x, y, z not all zero if and only if $f = 0$ for integers x, y, z, not all zero, since f is homogeneous.

Let then $\xi \neq 0$, and put $\eta = a(Z - XY)$, $\zeta = Y - aXZ$, so that

$$\begin{aligned} b\eta^2 - ab\zeta^2 &= b\big[a^2(Z^2 - 2XYZ + X^2Y^2) \\ &\qquad - a(Y^2 - 2aXYZ + a^2X^2Z^2)\big] \\ &= -ab\big[Y^2(1 - aX^2) - aZ^2(1 - aX^2)\big] \\ &= -a(1 - aX^2)(bY^2 - abZ^2) = -a\xi^2, \\ \delta\phi(\xi, \eta, \zeta) &\equiv a\xi^2 + b\eta^2 - ab\zeta^2 = 0, \end{aligned}$$

and $\phi(\xi, \eta, \zeta) = 0$ for $\xi \neq 0$. Hence again ϕ, and therefore also f, are zero forms, and we have proved the Dickson Theorem. The above proof is a rational proof holding for any field F so we have immediately the following result.

LEMMA 2. *If a ternary $f(x, y, z)$ with coefficients in F represents the associated quantity δ^{-1}, then f is a zero form.*

4. *Universal Ternaries over F.* We shall now prove the following theorem.

THEOREM 2. *A non-singular ternary quadratic form over F is universal over F if and only if it is a zero form.*

For let f be a zero form, so that $f(x, y, z) = 0$ for x, y, z not all zero and in F. Then

$$\delta\phi \equiv \psi(\xi, \eta, \zeta) \equiv a\xi^2 + b\eta^2 - ab\zeta^2 = 0$$

for ξ, η, ζ not all zero and in F. Let ρ be any quantity of F, $\sigma = \rho\delta$. If $\xi = 0$, then $b(\eta^2 - a\zeta^2) = 0$, whence $\eta^2 = a\zeta^2$, so that $\zeta\eta \neq 0$. Thus write $\xi_0 = \zeta\eta^{-1}$, from which $a\xi_0^2 = 1$. Put

$$X = 0, \qquad Y = \frac{\sigma + b^{-1}}{2}, \qquad Z = \frac{\sigma - b^{-1}}{2}\xi_0,$$

so that, since $1 = a\xi_0^2$,

$$\begin{aligned} 4\psi(X, Y, Z) &= b\left[(\sigma + b^{-1})^2 - (\sigma - b^{-1})^2 a\xi_0^2\right] \\ &= b\left[(\sigma + b^{-1})^2 - (\sigma - b^{-1})^2\right] \\ &= 4bb^{-1}\sigma = 4\sigma, \qquad \text{and} \qquad \psi = \sigma. \end{aligned}$$

Then $\phi = \delta^{-1}\sigma = \rho$ and hence $f = \rho$ for corresponding x, y, z in F.

Next let $\xi \neq 0$. Then $a + b(\eta\xi^{-1})^2 - ab(\zeta\xi^{-1})^2 = 0$, and if we write $\eta\xi^{-1} = a\zeta_0$, $\zeta\xi^{-1} = \eta_0$, we have $a + a^2b\zeta_0^2 - ab\eta_0^2 = 0$, $1 = b\eta_0^2 - ab\zeta_0^2$. Then put

$$X = \frac{\sigma + a^{-1}}{2}, \quad Y = \frac{\sigma - a^{-1}}{2}a\zeta_0, \quad Z = \frac{\sigma - a^{-1}}{2}\eta_0,$$

whence

$$\begin{aligned} 4\psi(X, Y, Z) &= a(\sigma + a^{-1})^2 + (ba^2\zeta_0^2 - ab\eta_0^2)(\sigma - a^{-1})^2 \\ &= a\left[(\sigma + a^{-1})^2 - (\sigma - a^{-1})^2\right] = 4aa^{-1}\sigma = 4\sigma, \\ \psi &= \sigma, \quad \phi = \delta^{-1}\sigma = \rho. \end{aligned}$$

Hence in this case also $f = \rho$ as desired, so that f is universal.

Conversely let f be universal. Then f represents δ^{-1} and, by Lemma 2, is a zero form. This proves* Theorem 2.

It is well known† that the determinant of the form $\phi(X, Y, Z)$ equivalent to f is h^2d, where h is the determinant of the transformation. Hence $-\alpha\beta\gamma = h^2d$, so that

$$\delta = \gamma(\alpha\beta)^{-1} = (\alpha\beta\gamma)(\alpha\beta)^{-2} = -\, dh^2(\alpha\beta)^{-2} = -\, dk^2,$$

where k is in F. Then

$$-\, df = \delta k^{-2}\phi = k^{-2}\psi(X, Y, Z) = \psi(\xi, \eta, \zeta),$$

for $X = k\xi \cdot X = k\eta \cdot Z = k\zeta$. Hence if f represents the negative of its determinant, the form $\psi = -df = (-d)^2$ represents d^2, and hence unity, and hence f is a zero form by Lemma 2. We may therefore replace Lemma 2 by the following statement.

THEOREM 3. *If f is a ternary with non-zero determinant d, then $-df(x, y, z) \equiv \psi(X, Y, Z) \equiv aX^2 + bY^2 - abZ^2$ for a suitable transformation. Also f is a universal zero form if and only if f represents $-d$.*

In particular the above Theorem 2 holds for the case where $F = R$, the field of all rational numbers. If, however, a is any rational number, then $a = b^{-2}c$, where b and c are integers. Obviously, if $f = a$ for rational x, y, z, then $f = c$ for rational x, y, z. Hence we have proved a partial converse to Dickson's theorem.

THEOREM 4. *A non-singular ternary quadratic form with integer coefficients is a zero form if and only if it represents all integers for rational values of its variables.*

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* It is evident that Theorem 2 is true if it can be proved for forms of type of $\psi(X, Y, Z) \equiv aX^2 + bY^2 - abZ^2$. If $(1, i, j, ij)$, $i^2 = a$, $j = b$, $ji = -ij$, is a generalized quaternion algebra over F, then for $ab \neq 0$, this algebra is either a division algebra or a total matric algebra. If $q = Xi + Yj + Zij$, then $q^2 = \psi(X, Y, Z)$. Hence, if ψ is a zero form, the algebra Q is not a division algebra and there exists a two-rowed matrix whose square is σ so that ψ represents σ. The converse of Theorem 2 is similarly proved. It is in fact this linear algebra theorem (which has long been known to me) which gave me an immediate proof of Theorem 2 as soon as I discovered the reduction given by (1)–(3).

† See Dickson, *Modern Algebraic Theories*, pp. 64–70.