ON IRREDUCIBLE SYSTEMS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

BY E. GOURIN

Given two irreducible algebraic manifolds, if the first is a proper sub-manifold of the second, the first is of lower dimensionality than the second.* We prove here an analogous result for systems of algebraic differential equations.

Terminology and notation will be as in Ritt's monograph, *Differential Equations from the Algebraic Standpoint.*† Let Σ_1 be a non-trivial closed irreducible system. Let the unknowns be $u_1, \cdots, u_q, y_1, \cdots, y_p$, with the u_i arbitrary unknowns. We prove the following theorem.

THEOREM. *If the manifold of the non-trivial closed irreducible system Σ_1 is a proper sub-manifold of the manifold of another such system Σ_2, then either Σ_2 has a set of arbitrary unknowns of which $u_1, \cdots, u_q$ form a proper subset, or $u_1, \cdots, u_q$ form a complete set of arbitrary unknowns of the system Σ_2 and, for these arbitrary unknowns, resolvents of Σ_1 are of lower order than those of Σ_2.*‡

Because Σ_2 holds Σ_1, and Σ_1 is closed, every form of Σ_2 is contained in Σ_1. Then, certainly, Σ_2 cannot contain a form involving the u_i alone. Otherwise Σ_1 would contain such a form. Consequently, there exists in Σ_2 a set of arbitrary unknowns of which $u_1, \cdots, u_q$ form a subset. This subset is either a proper subset or a full set of arbitrary unknowns of Σ_2.

Let us assume, then, that this latter condition is satisfied, that is, $u_1, \cdots, u_q$ form a complete set of arbitrary unknowns of both systems Σ_1 and Σ_2.

In order to construct a resolvent of Σ_2 we choose two forms G and Q, with G not in Σ_2 and free of the y_i, such that, for any two distinct solutions of Σ_2 with the same u_i, such that G does not vanish for these solutions, Q yields two distinct functions

* Van der Waerden, *Moderne Algebra*, vol. 2, p. 63.

† Published by this Society, 1932.

‡ We assume that the associated field contains a non-constant function. This involves no loss of generality.

of x. Having selected two such forms, we adjoin to Σ_2 the form $w-Q$, where w is a new unknown, and obtain a system, say Λ_2. It is clear, however, that the above described properties of the forms G and Q with respect to the system Σ_2 will remain undisturbed when tested with respect to the system Σ_1. Hence, the same forms G and Q may be utilized in the construction of the resolvent of Σ_1. Accordingly, we adjoin the same form $w-Q$ to Σ_1, obtaining a system Λ_1.

Let Ω_1 and Ω_2 be the two systems of forms in w, the u_i, and y_i, which vanish for all solutions of Λ_1 and Λ_2, respectively. The two systems Ω_1 and Ω_2 are closed and irreducible. Furthermore, any form of Ω_2 is contained in Ω_1.

We now list the unknowns in Ω_1 in the order $u_1, \cdots, u_q; w; y_1, \cdots, y_p$, and take a basic set for Ω_1

$$A, A_1, \cdots, A_p, \tag{1}$$

in which $w, y_1, \cdots, y_p$ are introduced in succession and in which A is algebraically irreducible. The equation $A=0$ is a resolvent of Ω_1. Each A_k is a form in w, the u_i, and y_k alone, is of order 0 in y_k, and of the first degree in y_k.

Similarly we proceed with Ω_2 and take a basic set

$$B, B_1, \cdots, B_p, \tag{2}$$

with B algebraically irreducible. Then $B=0$ is a resolvent of Ω_2 and each B_k is of the same structure as the corresponding A_k.

It is clear that A cannot be of higher order in w than B. Let us assume, then, that A and B are of the same order in w. Let I be the initial of A. Then, for an appropriate non-negative integer t, we have the identity

$$I^tB = AC + D, \tag{3}$$

where C and D are forms in w and the u_i, D being of lower rank in w than A. Since D vanishes for every solution of Ω_1, D is contained in Ω_1. Hence $D\equiv 0$.

It follows, therefore, from (3) that I^tB is divisible by A. Because A is algebraically irreducible, and I and A are relatively prime, B must be divisible by A. Let $B=A\cdot K$. Then K is a function of x, the form B being algebraically irreducible. This implies that in (1) we may replace A by B, because, in choosing this basic set, we may select for A among the forms of Ω_1 involving w and the u_i alone, any algebraically irreducible form which is of the least rank in w. Such a form, however, was found to be a member of Ω_2 and hence may be identified with the form B.

B_1 is contained in Ω_1. It is reduced with respect to B and, of all such forms in Ω_1 in the u_i, w, and y_1, it certainly has a lowest rank. Consequently we may replace A_1, in (1), by B_1. Continuing, we find that (2) is a basic set for Ω_1. Then Σ_1 and Σ_2 are identical. This contradiction proves that A is of lower order in w than B and establishes our theorem.

Columbia University

AXIOM C OF HAUSDORFF AND THE PROPERTY OF BOREL-LEBESGUE*

BY SELBY ROBINSON†

1. *Introduction.* This is a study in an abstract space (P, K) of the Hausdorff‡ property C which may be expressed in the form *the interior of every set is an open set.* A point p of the space P is interior to a set V, if p is a point of V and is not a K-point (point of accumulation, limit point) of any subset of $C(V)$. An open set is one all of whose points are interior points. We say that space (P, K) has property B of Hausdorff if and only if any point p which is interior to each of two sets is interior to their logical product; we shall designate as the open set B property, the weaker property: the product of two open sets is an open set.§ By the Hausdorff property D we shall mean that any two points are respectively interior to sets which are disjoined, while in the open set D property the points are required to be in disjoined open sets. The Borel and Borel-Lebesgue properties take three non-equivalent forms in spaces not having property C. These three forms coincide if property C is present as do the two forms of property B and of property D. In §3 we consider three

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† National Research Fellow.

‡ F. Hausdorff, *Grundzüge der Mengenlehre*, first edition, 1914, p. 213.

§ Chittenden chose the open set B property as the one to designate as the Hausdorff B property. See Transactions of this Society, vol. 31 (1929), p. 315.