AXIOMATIC DEFINITIONS OF PERFECTLY SEPARABLE METRIC SPACES*

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1. *Introduction.* Among topological spaces, perfectly separable metric spaces (abbreviated PSM-spaces) play a major† role; therefore their axiomatic definition is a problem of peculiar interest. We shall discuss axiomatic definitions of PSM-spaces.

A *topological space* is‡ a class $\mathfrak{S}$ of "points" x, and an operation K assigning to any set S of points of $\mathfrak{S}$, a "derived" set $K(S)$. In practice, K is defined in one of three ways:

I. By ascribing to every point x of $\mathfrak{S}$ a family of "neighborhoods of x," denoted by V_x with variable superscript. Here $K(S) \supset x$, if and only if no $(V_x - x)\cdot S$ is vacuous.

II. By ascribing to every ordered pair (x, y) of points of $\mathfrak{S}$, a number $\overline{xy}\cdot\overline{xy} > 0$ if $x \neq y$; $\overline{xx} = 0$. Here $K(S) \supset x$, if and only if g.l.b. $_{S \supset x \neq y}\, \overline{xy} = 0$.

III. By ascribing to every point x of $\mathfrak{S}$ a family of infinite sequences $\{x_k\}$ of distinct points of $\mathfrak{S}$, which "converge to x" (abbreviated $x_k \rightarrow x$). Here $K(S) \supset x$, if and only if we can find a sequence in S converging to x.

Spaces defined as in I are known as spaces (V); we shall call spaces defined as in II spaces (Q), and those defined as in III, spaces (Λ).

A system of neighborhood axioms, such that any space (V) satisfying it is a PSM-space, has been deduced by Hausdorff-Urysohn-Tychonoff.§ With slight modifications, the axioms are satisfied by every representation of a PSM-space as a space (V). It is therefore futile to attempt to weaken them except as regards interdependence.

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* Adapted from a Harvard Undergraduate Honors Thesis.

† For they are precisely the class used in function theory. See Mathematische Annalen, vol. 92 (1924), p. 302.

‡ Fréchet, *Les Espaces Abstraits*, 1928, p. 167.

§ See Hausdorff's *Grundzüge der Mengenlehre*, 1914, Chaps. 7–8; Urysohn, *Zum Metrisationsproblem*, Mathematische Annalen, vol. 94 (1925), p. 309; Tychonoff, *Über ein Satz*, etc., Mathematische Annalen, vol. 95 (1925), p. 139.

Systems of distance axioms, all spaces (Q) that satisfy which are metric spaces, have been deduced by Chittenden-Niemytski-Wilson.* We shall proceed by another line to find a system of distance axioms, effectively weaker under the additional hypothesis of perfect separability—satisfied in fact by every representation of a PSM-space as a space (Q). No system of convergence axioms, all spaces (Λ) that satisfy which are PSM-spaces, has, I believe, been set up hitherto. We shall construct such a system, showing in what sense it is necessary as well as sufficient.

2. *Spaces* (Λ) *and Spaces* (Q). The class of spaces (Λ) is quite evidently topologically equivalent† (abbreviated TE) to the class of spaces (V) in which, if $K(S) \supset x$, S contains an enumerable subset S' such that $K(S') \supset x$. Suppose that in addition we can so choose S' that $\{y_k\} \subset S'$ implies $K(\{y_k\}) \supset x$. We can make the class of such sets S' our fundamental class of convergent sequences. We then have a space (Λ) in which

(A) *If* $x_k \to x$, *and* $\{y_k\} \subset \{x_k\}$, *then* $y_k \to x$. *Conversely if, given* $\{y_k\} \subset \{x_k\}$, *we can so choose* $\{z_k\} \subset \{y_k\}$ *that* $z_k \to x$, *then* $x_k \to x$.

THEOREM 1. *Let* $(\mathfrak{S}, K)$ *be a space* (Λ) *satisfying*

(B) *If* $x_k \to x$, *and* $\{y_k\} \subset \{x_k\}$, *then we can so choose* $\{z_k\} \subset \{y_k\}$ *that* $z_k \to x$.

Then $(\mathfrak{S}, K)$ *has a* TE *representation as a space* (Λ) *satisfying* (A).

For add to the primitive families of convergent sequences of $(\mathfrak{S}, K)$, all sequences satisfying the second half of (A). Then K will be unchanged, yet (A) will be satisfied by the augmented families of convergent sequences. It is noteworthy that any further augmentation either contradicts (B) or changes K.

A space is called *locally separable* if and only if it is TE to a space (V) in which the family of neighborhoods ascribed to any x can be enumerated: $V_x^1, V_x^2, V_x^3, \cdots$. Every space (Q) is locally separable; let $V_x^n \supset y$ if and only if $\overline{xy} < 1/n$. Every space

* For latest results see Niemytski's *On the third axiom of metric space*, Transactions of this Society, vol. 29 (1927), p. 666; also W. A. Wilson's two articles in the American Journal, vol. 53 (1931).

† Two topological spaces are called "topologically equivalent" if and only if a (1, 1)-correspondence which preserves K can be established between their points.

(Λ) is* a space (V), but not necessarily a locally separable one.

Let ($\mathfrak{S}$, K) be a locally separable space (Λ), and define a pseudo-distance $\widetilde{xy}$ as follows: $\widetilde{xx}=0$; and $\widetilde{xy}=1/n$ $[x\neq y]$, if V_x^n is the first V_x in order of enumeration not† containing y. If $\text{g.l.b.}_{S\supset y\neq x}\widetilde{xy}=0$, then plainly $K(S)\supset x$; i.e., given n, $y\subset S$ exists contained in at least the first n neighborhood V_x^k. Suppose $K(S)\supset x$, yet $\text{g.l.b.}_{S\supset y\neq x}\widetilde{xy}>0$. Hence $S\supset x_k\rightarrow x$, yet $\text{g.l.b.}\,\widetilde{xx_k}>0$; therefore that some V_x^n excludes an infinity of the x_k. Extracting the subsequence not in V_x^n, we obtain an exception to (A).

Conversely, in a space (Q), let us write $x_k\rightarrow x$ if and only if $\overline{xx_k}\rightarrow 0$. It follows from the theory of limits, that $K(S)\supset y$ if and only if an infinite sequence $\{x_k\}$ exists in S, satisfying $x_k\rightarrow y$. Moreover the pseudo-convergences just defined satisfy (A). We may therefore state the following theorem.

THEOREM 2. *The class of locally separable spaces* (Λ) *satisfying* (A) *is* TE *to the class of spaces* (Q). *Moreover* $\overline{xx_k}\rightarrow 0$ *is equivalent to* $x_k\rightarrow x$.

3. *Locally Separable Hausdorff Spaces.* Consider ($\mathfrak{S}$, K), any space (Q) satisfying the two conditions

(α) If $\overline{xz_i}+\overline{yz_i}\rightarrow 0$, then $x=y$.

(β) If $\overline{xx_i}\rightarrow 0$, and $\overline{x_ix_j^i}\rightarrow 0$ identically in i, then $N(i)$ exists so large that $j(i)>N(i)$ implies $\overline{xx_{j(i)}}\rightarrow 0$.

LEMMA. $K\{K(S)\}\subset K(S)$ *for any set* S *in* ($\mathfrak{S}$, K).

For suppose $x\subset K\{K(S)\}$; we can evidently so choose $x_i\subset K(S)$ that $\overline{xx_i}\rightarrow 0$. But since $x_i\subset K(S)$, we can so choose $x_j^i\neq x$ in S that $\overline{x_ix_j^i}\rightarrow 0$, irrespective of i. Hence by (β), $\text{g.l.b.}_{S\supset x_j^i\neq x}\overline{xx_j^i}=0$, and $x\subset K(S)$.

Let us define V_x^n $[x\subset\mathfrak{S};\ n=1,\ 2,\ 3,\cdots]$ as the set of all points y which satisfy $\overline{xy}>1/n$; let $W_x^n=V_x^n-K(\mathfrak{S}-V_x^n)$. Then $V_x^n\supset x$, but x is not $\subset K(\mathfrak{S}-V_x^n)$, so that the following statement will be true.

(a) To every x corresponds a W_x^n, and every W_x^n contains x. Moreover the theory of limits shows us that in any space (Q), $K(S+T)=K(S)+K(T)$. It follows by this and the Lemma that

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* Since $S\supset S'$ implies $K(S)\supset K(S')$; see *Les Espaces Abstraits*, p. 179.

† If $x\neq y$, *some* $V_x^n\supset y$; otherwise $K(y)\supset x$, and since y contains no infinite sequence, this is impossible.

$$(\phi) \qquad K(\mathfrak{S} - W_x{}^i) = K(\mathfrak{S} - V_x{}^n) + K\{K(\mathfrak{S} - V_x{}^n)\}$$
$$= K(\mathfrak{S} - V_x{}^n) \subset \mathfrak{S} - W_x{}^n.$$

Now suppose that some $(W_x^n - x)\cdot S$ is vacuous, so that $(\mathfrak{S} - W_x^n) + x \supset S$. It follows that $K(\mathfrak{S} - W_x^n) + K(x) \supset K(S)$; hence, using (ϕ) and the fact that $K(x)$ is null, we see that $(\mathfrak{S} - W_x^n) \supset K(S)$, and, by (a), $K(S)$ is not $\supset x$. Conversely, if no $(W_x^n - x)\cdot S$ is vacuous, certainly no $(V_x^n - x)\cdot S$ is vacuous, and $K(S) \supset x$. That is, the W_x^n are neighborhoods of x in the sense of I.

(b) Let W_x^i and W_x^j be given. Then $W_x^{i+j} \subset W_x^i \cdot W_x^j$ exists.

(c) If $y \subset W_x^n$, then y is not $\subset (\mathfrak{S} - W_x^n)$, whence by (ϕ), we have y is not $\subset K(\mathfrak{S} - W_x^n)$. Therefore W_y^n exists, making $W_y^n \cdot (\mathfrak{S} - W_x^n)$ null, and hence contained in W_x^n.

(d) Let $x \neq y$ be given. Then W_x^n and W_y^n exist without common point; otherwise $z_n \subset W_x^n$, W_y^n exists for every n, and $\overline{xz_n} + \overline{yz_n} \to 0$, contradicting (α).

Conversely, setting up a pseudo-distance just as in the case of locally separable spaces (Λ), we obtain a TE space (Q) from any space (V) which is locally separable and satisfies (a) and (b). Further, we can obtain (α) from (d) and (β) from (c). Therefore we may state the following result.

THEOREM 3. *The class of spaces* (Q) *satisfying* (α) *and* (β) *is* TE *to the class of locally separable Hausdorff spaces.*

4. *Two Further Conditions.* A topological space is said to be *regular** if and only if to every point x and every closed set S not containing x correspond disjoint open sets $U \supset x$ and $V \supset S$. Suppose $(\mathfrak{S}, K)$ is a space (Q) satisfying (α) and (β) but not regular, and let x and S defined as above be contained in no two disjoint open sets. Let finally W be defined as in §3.

Every W_x^i must have either a point or a limit point lying in S; otherwise we could enclose each point of S in an open set disjoint to W_x^i, and the sum of these open sets would be an open set containing S and disjoint to $W_x^i \supset x$. Since $K(S)$ is not $\supset x$, $W_x^i \cdot S$ is vacuous for $i > M(S, x)$, and to $i > M(S, x)$ correspond $x_j^i \subset W_x^i$ such that $x_j^i \to x_i \subset S$. But since x is not $\supset S \supset K(S)$, $\text{g.l.b.}_{S \supset x_j^i} \overline{xx_j^i} > 0$. That is, if we set $N(i) = 0$, there exists an exception to this statement:

* E. W. Chittenden, this Bulletin, vol. 33 (1927), p. 20. The term is due to Alexandroff and Urysohn, Mathematische Annalen, vol. 92 (1924), p. 263.

(β') If $N(i)$ exists so large that $j(i)>N(i)$ implies $\overline{xx^i_{j(i)}}\rightarrow 0$, and $\overline{x_ix^i_j}\rightarrow 0$ identically in i, then $\overline{xx_i}\rightarrow 0$.

We have therefore proved the following fact.

THEOREM 4. *Every space* (Q) *satisfying* (α), (β), *and* (β') *is* TE *to a regular locally separable Hausdorff space.**

We shall call a space *perfectly separable* if and only if we have:

(γ) There exist open sets N_1, N_2, N_3, $\cdots$, of which any open set is the sum of a subclass.

THEOREM 5. *If a Hausdorff space satisfies* (γ), *it is perfectly separable in the sense of Hausdorff, that is, we can take the* N_i *for neighborhoods; and conversely.*

Let N_i be a neighborhood of x if and only if $N_i\supset x$, and observe (1) that since each N_i is open, it contains at least one neighborhood of every point contained in itself; (2) that since by (c) every V_x is open, $V_x=\sum N_i$ summed over a suitable range of i,—whence since in addition by (A) $V_x\supset x$, it follows that $V_x\supset N_i\supset x$ is satisfied by at least one N_i. It results from (1) and (2) that the N_i define K correctly, while (A) – (D) can be easily deduced from (1), (2), and the fact that the N_i are open sets containing the points of which they are neighborhoods. To prove the converse, merely set the V_x as N_i.

THEOREM 6. *If the derived sets of a space* (Λ) *are closed, and it satisfies* (γ), *it is locally separable.*

For in this case the interior points of neighborhoods of a point form an equivalent† system of neighborhoods and are open; from this point we can proceed as in the proof of the first part of Theorem 5.

If we note that $\overline{xx_i}\rightarrow 0$ means simply that $k_i<k_{i+1}$ implies $K(\sum_1^\infty x_{k_i})\supset x$, we see immediately that conditions (α)–(γ) are topologically invariant. Likewise, since under any metric‡ representation $\overline{xz_i}+\overline{yz_i}\geqq\overline{xy}$, while we can so choose $N(i)$ that $\overline{x_ix^i_j}<1/i$, whence $|\overline{xx_i}-\overline{xx^i_j}|<1/i$, for $j>N(i)$, we see that (α)–(γ) hold under the metric representation of a PSM-space.

* The converse is also true.

† *Les Espaces Abstraits*, p. 173.

‡ A metric representation is a representation as a space (Q) in which a. $\overline{xy}=\overline{yx}$, b. $\overline{xy}+\overline{yz}\geqq\overline{xz}$.

Hence (α)-(γ) hold under any representation of a PSM-space as a space (Q). However, Tychonoff† has proved that a necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be regular. Applying Theorems 4 and 5, we may therefore state the following conclusion.

THEOREM 7. *A necessary and sufficient condition that a space* (Q) *be a* PSM-*space is that* (α), (β), (β'), *and* (γ) *hold.*

5. *Conclusions.* Inasmuch as every space (Q) is locally separable, and $\aleph^2 = \aleph$, it might be conjectured that (γ) was replaceable by the following neater condition:

(γ') $\quad \Gamma = x_1 + x_2 + x_3 + \cdots$ exists in $\mathfrak{S}$, and $K(\Gamma) = \mathfrak{S}$.

THEOREM 8. *There exists a space* (Q) *which is not a* PSM-*space, yet which satisfies* (α), (β), (β'), *and* (γ').

For let $\mathfrak{S}$ consist of all points $x\,[0<x<1]$, and let $\overline{xy} = (x-y)$ or 1, according as $x \geqq y$ or $x<y$. That $(\mathfrak{S}, K)$ satisfies the hypotheses is fairly obvious; the only difficulty is to show that it is not perfectly separable. To show this, remark that *every* V_{x_0} contains a set $x_0 > x > x_0 - \epsilon$, and some $V_{x_0}^*$ excludes the entire set $x < x_0$ as well. Now $V_x^* = V_y^*$ only if $x = y$. Hence the neighborhoods are certainly not enumerable.

Let us disprove in the same way a second conceivable conjecture. Let $\mathfrak{S}$ consist of two classes of points, points p_k and points $q_k\,[0 = k = 1]$. Let $\overline{q_i q_j} = \overline{q_i p_j} = \overline{p_k q_k} = \overline{q_k p_k} = 1$, and $\overline{p_i p_j} = \overline{p_i q_j} = |i-j|, [i \neq j]$. It is easy to verify, using Theorem 4, that the space is a compact regular locally separable Hausdorff space. It is not perfectly separable, as it contains more than countable isolated points. Therefore it is not metric; every compact metric space is‡ perfectly separable. Hence we have the following fact.

THEOREM 9. *There exists a compact regular locally separable Hausdorff space which is not metric.*

By simple ad hoc constructions we can show the independence of (α), $(\beta)+(\beta')$, and (γ) as conditions on a space (Q); the last

† Mathematische Annalen, vol. 95 (1925), p. 139.

‡ See Urysohn, *Zum Metrisationsproblem*, Mathematische Annalen, vol. 94 (1925), p. 309.

is incidentally proved by Theorem 8. From this and Theorem 7 we have the following result.

THEOREM 10. *The following distance axioms determine which spaces* (*Q*) *are* PSM-*spaces*:

(α) *If* $\overline{xz}_i+\overline{yz}_i\to 0$, *then* $x=y$.

(β'') *Let* $\overline{x_ix_j^i}\to 0$ *identically in* i. *Then* $\overline{xx}_i\to 0$ *if and only if* $N(i)$ *exists, so large that* $j(i)>N(i)$ *implies* $\overline{xx}^i_{j(i)}\to 0$.

(γ) *There exist open sets*† $N_1, N_2, N_3, \cdots$, *of which any open set is the sum of a subclass.*

Employing Theorem 2 to replace (α) and (β'') by equivalent axioms, and suppressing the hypothesis of local separability as being implied in the others (see Theorem 6), we obtain the following theorem.

THEOREM 11. *Taken together with* (γ), *the following convergence axioms determine which spaces* (Λ) *satisfying* (A) *are* PSM-*spaces*:

(B) *If* $z_i\to x$, *and* $z_i\to y$, *then* $x=y$.

(C) *Let* $x_j^i\to x_i$ *identically in* i. *Then* $x_i\to x$ *if and only if* $N(i)$ *exists, so large that* $j(i)>N(i)$ *implies* $x^i_{j(i)}\to x$.

Finally, let ($\mathfrak{S}$, K) be any *topological space*. Let us make the *definition*, $\{x_k\}\to x$ if and only if $\{y_k\}\subset\{x_k\}$—all points of the same infinite sequence being assumed distinct—implies $K(\{y_k\})\supset x$. Then, since (A′) and the new definition of convergence imply (A), we may state the following result.

THEOREM 12. *The following convergence axioms determine which topological spaces are* PSM-*spaces*:

(A′) $K(S)\supset x$ *if and only if* $\{x_k\}\subset S$ *exists, such that* $\{x_k\}\to x$.

(B) *If* $z_i\to x$, *and* $z_i\to y$, *then* $x=y$.

(C) *Let* $x_j^i\to x_i$ *identically in* i. *Then* $x_i\to x$ *if and only if* $N(i)$ *exists, so large that* $j(i)>N(i)$ *implies* $x^i_{j(i)}\to x$.

(γ) *There exist open sets*‡ $N_1, N_2, N_3, \cdots$, *of which any open set is the sum of a subclass.*

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† An open set is a set S such that, given $x\subset S$, g.l.b.$_{y\subset S}\overline{xy}>0$.

‡ An open set is a set S such that, if $\{x_x\}\subset\mathfrak{S}-S$, and $x\subset S$, then $\{x_x\}$ does not $\to x$