THESES ON CALCULUS OF VARIATIONS, 1931–32

Contributions to the Calculus of Variations, 1931–1932. Theses submitted to the Department of Mathematics at the University of Chicago. University of Chicago Press, February, 1933. 523 pp.

The present volume, uniform in appearance with the one published under similar title in 1930 (reviewed in this Bulletin, vol. 38 (1932), p. 617), contains eight doctoral dissertations, one master's thesis (No. 10 below), and an article (No. 6 below), written in pursuit of unalloyed scientific interest, rather than "submitted in candidacy for" a degree. It is gratifying to observe that the reception accorded the frankly experimental "contributions" of 1930 was sufficiently enthusiastic to justify a continuation of the venture. In size, in proof reading, in general appearance, this volume surpasses its predecessor; in significance of content it also marks an advance. In the preface, signed by Professors Bliss and Graves, under whose direction the theses contained in the volume were written, it is pointed out that one of the advantages of this form of publication of doctoral theses is that it makes it possible to give "the candidate for the Ph.D. degree as much freedom in his writing as is consistent with mathematical accuracy and clearness of style, even if the result is more verbose than could reasonably be approved for publication in a journal." It is therefore to be understood that the volume has not been edited, that each author is responsible for his own paper only, and that the book is to be looked upon as a collection of individual papers, linked together only in so far as their subjects are related, and by the incidental fact that they all represent work done by the Department of Mathematics of the University of Chicago.

The book consists of ten parts:

1. *Edge conditions for multiple integrals in the calculus of variations*, by J. E. Powell, (pp. 1–62).

2. *The Euler-Lagrange multiplier rule for double integrals*, by M. Coral, (pp. 63–94).

3. *The condition of Mayer for discontinuous solutions of the Lagrange problem*, by R. A. Hefner, (pp. 95–130).

4. *A problem in the calculus of variations suggested by a problem in economics*, by H. H. Pixley, (pp. 131–190).

5. *Functions of lines and the calculus of variations*, by R. G. Sanger, (pp. 191–294).

6. *Sufficient conditions for a problem of Mayer in the calculus of variations*, by G. A. Bliss and M. R. Hesteness, (pp. 295–338).

7. *Sufficient conditions for the general problem of Mayer with variable endpoints*, by M. R. Hestenes, (pp. 339–360).

8. *The problem of Bolza and its accessory boundary value problem*, by Kuen-Sen Hu, (pp. 361–444).

9. *Jacobi's condition for multiple integral problems of the calculus of variations*, by A. W. Raab, (pp. 445–474).

10. *A history of the classical isoperimetric problem*, by T. I. Porter, (pp. 475–523).

To begin with an account of the historical papers (Nos. 5 and 10), they review the literature on the subjects mentioned in their titles in a coherent, but not a critical, way. Dr. Sanger presents in two successive chapters the definitions for the derivative and the differential of a functional, as given by Volterra and Fréchet, respectively, and as developed and modified by various later writers; the third chapter deals with the applications of these concepts to the calculus of variations; it is followed by a bibliography consisting of eighty-three titles. No sharp distinction is made between what, using E. H. Moore's terminology, might be called functions on C to A, that is, functionals, which associate a real number with every function of a certain class, and functions on C_1 to C_2, that is, functional operations by means of which a function is made to correspond to a function. The work of Mandelbrojt, for instance, presented on pages 238–239 belongs in the latter field; most of the other work to the former. There is ample room for differences of opinion as to the relative importance to be attached to the various contributions dealt with in a historical survey of this character; it is not easy to obtain an adequate perspective over a field of which a great deal is still in the early stages of development and of which the importance can perhaps be appreciated only after the lapse of years.

The subject treated by Mr. Porter, more restricted in character, and one which has received complete and detailed study, lends itself more readily to a critical discussion. The problem with which nearly the entire paper is concerned is that of proving that among all closed plane curves of given length the circle encloses maximum area. An interesting list of seventy-five titles closes the paper. That the literature relating to the extensions and generalizations of this problem is still growing is shown for example by Abstract 114, published on page 208 of volume 39 of this Bulletin.

Rather apart from the rest of the volume in character and interest is paper No. 4, concerned primarily with conditions for a maximum of the integrals $\int f(x, y, y')\, dx$ and $\int f(x, y, y', y'')\, dx$, in the first of which y and y' are allowed to possess a finite number of finite discontinuities, while in the second y is supposed to be continuous, but y' and y'' may have a finite number of finite discontinuities. The conclusions reached by Dr. Pixley are that, in general, problems of this character, under his unrestricted conditions of admissibility, do not possess a solution. It would therefore be of special interest to establish a connection between this dissertation and the paper of Razmadzé (Mathematische Annalen, vol. 94 (1925), p. 1) in which an analogous problem with different conditions and different conclusions is discussed.

In an introductory section certain problems in economics are presented which lead to such problems of the calculus of variations as are treated in this paper. It must be confessed that these questions of the calculus of variations are of quite sufficient interest in themselves, and that the importance of the monopoly problems to which they are to be applied escapes the reviewer. It is not clear what is the economic significance of the function $2f$ introduced on page 185, or of the conditions imposed on the constants in this function in order to obtain a non-vacuous result. It is to be hoped that the conclusions of this paper are not to be interpreted as an indication that most problems in economics when formulated rigorously fail to possess solutions.

Of the seven remaining parts, three (Nos. 1, 2, 9) are concerned with multiple integrals and four (Nos. 3, 6, 7, 8), with the problems of Lagrange, Mayer, and Bolza. We shall briefly discuss each of these groups.

Dr. Powell considers multiple integrals of the form $\int_S f(x, y, p)\,dx$ in which x and y are respectively n- and m-partite variables, and p is the set of partial derivatives of y with respect to x. The admissible manifolds are those which are defined by a set of functions $y(x)$ continuous over a bounded, closed, connected region S of x-space, which consists of a finite number of cells in which the partial derivatives of y are continuous. The vanishing of the first variation leads to the Euler-Lagrange equations, and to a generalization of the Erdmann-Weierstrass corner condition along the edges of the manifold, that is, the curves corresponding to the boundaries of the cells. Analogous *edge conditions* are then obtained for the related isoperimetric problem, in which a fixed value is assigned to a second integral of the same type as the one which is to be minimized. After a brief section dealing with a discontinuous integrand f, the edge condition is obtained once more by means of an extension to multiple integrals of Mason's lemma for double integrals under slightly more general hypotheses.

The extension of the Euler-Lagrange multiplier rule to problems involving multiple integrals and partial differential equations has progressed very slowly. A paper by Gross (1916) appears to be the only one in which the problem was materially advanced. In his contribution to the volume under review, Dr. Coral carries the work a considerable distance forward. The first chapter gives a simple treatment of the restricted problem formerly treated by Gross; the second chapter applies the same method to the solution of the problem of minimizing the double integral $\iint f(x, y, z_i, p_i, q_i)\,dx\,dy$, $(i=1, 2)$, by means of functions of a certain class which are moreover required to satisfy the differential equation $\phi(x, y, z_i, p_i, q_i)=0$. The new method developed by the author consists in reducing the given problem to an associated Lagrange problem involving only simple integrals.

In the last paper of this group, Dr. Raab applies Bliss' method for the treatment of second-order conditions to a multiple integral of the same form as the one for which edge conditions were derived by Dr. Powell. In this way he obtains as the analog of Jacobi's equation for the simple integral a system of linear partial differential equations of the second order of elliptic type. As a first step towards obtaining an analog of Jacobi's condition for this problem a result is obtained which, in the main, amounts to the following: If for a minimizing manifold which satisfies Legendre's condition in the stronger form, the Jacobi system of equations possesses a solution of class C'' which vanishes along a Green's manifold M within the domain of integration, then the corresponding first partial derivatives must all vanish on M. A Green's manifold is understood to be a closed $(n-1)$-dimensional manifold which bounds a portion of x-space, and for which the extension of Green's lemma is valid. A corresponding result is obtained for the multiple integral in parametric form.

Of the last group of papers in this volume, two have appeared in volume 35 of the Transactions of this Society, namely, No. 6 (pp. 305–327) and No. 7 (pp. 479–491). A discussion of their contents is therefore not in order here; suffice it to say that they complete and simplify in an important respect the recent work done on the problems of Mayer and Bolza by Morse and Bliss.

Paper No. 3 extends to the Lagrange problem in parametric form the results obtained by Graves (American Journal of Mathematics, vol. 52 (1930)) for discontinuous solutions of problems in n-space. Dr. Hefner's paper follows closely not only the methods used by Graves, but also the language used by that author. The most important result obtained is the extension of the Mayer condition to extremaloids, that is, to sets of functions $y_i(x)$, $\lambda_\alpha(x)$, of which the latter are continuous except possibly at corners, the former continuous, while together they satisfy the Euler equations and the corner conditions. The definition of *conjugate points* introduced by Graves is carried over to this problem and it is shown that these points are obtained for values of the parameter which cause a certain determinant to vanish or to change sign.

In paper No. 8, Dr. Hu deals with the problem of Bolza in essentially the same form as was used by Morse (American Journal of Mathematics, vol. 53 (1931)). He gives, however, a new formulation of the *accessory problem* in terms of a function $I_2(\xi, \eta)$ involving the end values of the variations as well as an integral connected with the second variation, a system of linear differential equations, end conditions, and the additional condition

$$\int_{x_1}^{x_2}(\xi_1^2+\xi_2^2+\eta_i\eta_i)dx=1.$$

This accessory problem is shown to be equivalent to a boundary value problem involving a parameter σ. It is in terms of this boundary value problem that the principal results of the paper are obtained, namely, a necessary and sufficient condition for a permanent sign of $I_2(\xi, \eta)$ for all variations which satisfy the conditions of the accessory problem, and a set of sufficient conditions. Further study of the boundary value problem and the related minimum problem lead to oscillation and comparison theorems, stated in terms of the distribution of the focal points of the manifold determined by the end conditions. This paper represents a valuable contribution to the study of the interrelations between the field of boundary value problems and the calculus of variations.

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