A PLANE ELLIPTIC CURVE OF ORDER $4k+2$, WITH SINGULARITIES ALL REAL AND DISTINCT, AND AUTOPOLAR BY $4k+4$ CONICS

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Comparatively few curves are known which are their own polar reciprocals (that is, autopolar) by a finite number of conic sections. The so-called *W*-curves of Klein and Lie are autopolar by a single infinity of conics, and these are the only curves having this property,* but these rational curves are highly specialized in that all their point and line singularities are concentrated at two places of the loci. Appell† has shown how to find all curves which are autopolar by one given conic. Haskell‡ has shown how Appell's method can be used to derive curves which are autopolar by a *finite* number of conics. This latter device, though sound in principle, involves severe algebraic difficulties in the very magnitude of the degree of the equations to be treated. Consequently only a very few curves whose singular elements are distinct are known to be autopolar by more than one conic. The quartic§ with two cusps and a node is autopolar by two conics. The quintic with five cusps is autopolar by one, two, or six conics, the number of polarizing conics depending upon the symmetry of the quintic. The completely|| symmetric quintic with three cusps and three nodes is autopolar by four conics; if the nodes and cusps unite to form three rhamphoid cusps, the quintic is still autopolar by three conics. The rational¶ quintic of class five in general is not autopolar. Beyond the fifth order only a few** isolated special cases of autopolar curves have been published. In the present and succeeding papers the writer proposes to show that there exist rational and

* Fouret, Bulletin de la Société Philomathique, 1877.

† Appell, Nouvelles Annales, (3), vol. 13 (1894).

‡ Haskell, Proceedings of the Toronto Congress, vol. I, 1924, pp. 715–717.

§ Wear, American Journal of Mathematics, vol. 42, pp. 97–118; note also American Journal of Mathematics, vol. 51, pp. 482–490.

|| Duncan, University of California Publications, 1928.

¶ Swinford, University of California Publications, 1929.

** Duncan, this Bulletin, vol. 39 (1933), pp. 589–592.

elliptic curves of every order above five having their singular elements all distinct and which are autopolar by a finite number of conics, the number of polarities actually increasing with the order of the special types considered. Moreover these curves can be sketched approximately with astonishing ease as the sequel will show. In the present paper only elliptic curves of order $4k+2$ are treated. The equations and polarities are exhibited for the two curves of lowest degree, namely six and ten.

The following discussion indicates that

$$(1) \qquad \rho^{2(2k+1)} \sin^2 (2k+1)\theta - (\rho^2 - 1)^3 \prod_{i=1}^{k-1} (\rho^2 - \alpha_i^2)^2 = 0,$$

$$(1 < \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1}),$$

is the equation in polar coordinates of a curve of order $4k+2$, elliptic, with singularities all distinct, and autopolar by $4k+4$ conics. The constants α_i^2 are the unique set of positive constants a_i which cause the equation

$$(2) \qquad x^{2k+1} - (x-1)^3 \prod_{i=1}^{k-1} (x - a_i)^2 = 0$$

to have $k-1$ double roots. The uniqueness of the set of a_i is shown below. One obtains (2) from (1) by setting

$$\rho^2 = x, \qquad \alpha_i^2 = a_i, \qquad \theta = \pi/(4k+2).$$

Observe that (1) represents a locus having the following properties.

(i) It is invariant under rotations about the origin through the angles $\lambda\pi/(2k+1)$, $(\lambda = 1, 2, \cdots, 4k+1)$.

(ii) On the line $\theta = 0$ it has cusps at $\rho = \pm 1$ and crunodes at $\rho = \pm\alpha_i$.

(iii) On the line $\theta = \pi/(4k+2)$ it has a biflecnode at infinity and $2(k-1)$ crunodes corresponding to square roots of a_i in (2).

(iv) The locus has $4k+2$ cusps and $(2k+1)(4k-3)$ nodes, the requisite number for an elliptic self-dual curve of order $4k+2$.

(v) The nodes at infinity are biflecnodes whose inflexional tangents are perpendicular to cuspidal tangents.

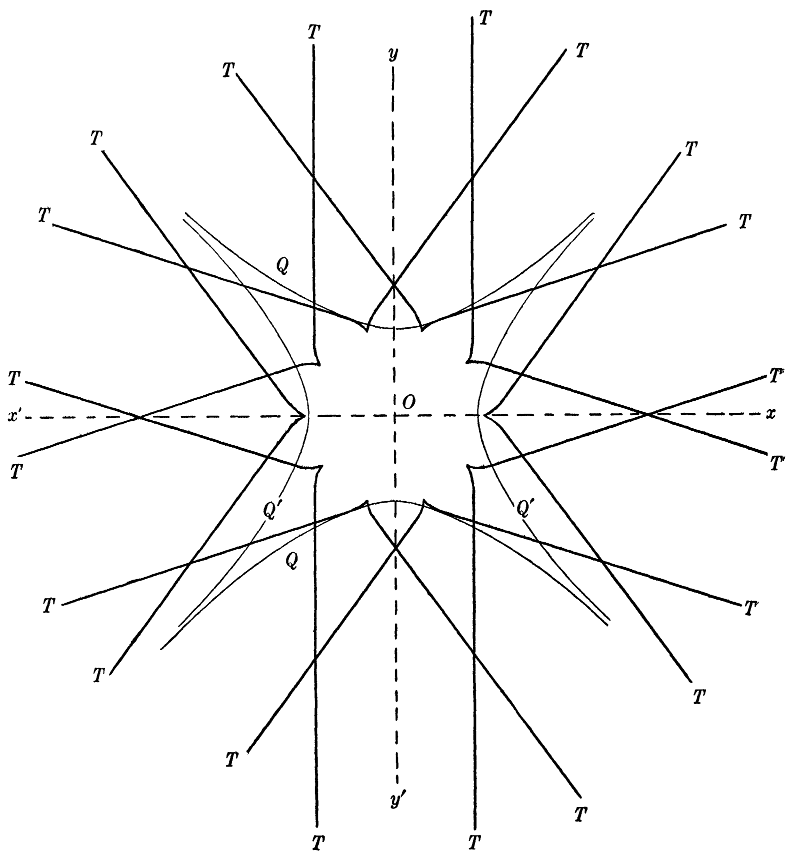
The uniqueness of the a_i in (2) is to be noted geometrically as follows. In $y = x^{2k+1} - (x-1)^3 \Pi_{i=1}^{k-1}(x-a_i)^2$, let $y_1 = x^{2k+1}$, $y_2 = (x-1)^3 \Pi_{i=1}^{k-1}(x-a_i)^2$, $y^{(1)} = (x-a_1)^2$, $y^{(2)} = (x-a_2)^2, \cdots,$

$y^{(k-1)}=(x-a_{k-1})^2$. Now for $1<a_1<a_2 \cdots <a_{k-1}$, $y^{(i)}$ represent $k-1$ distinct parabolas with vertices on the x-axis and opening upward; hence y_2 represents a curve having ordinary contacts with the x-axis at $x=a_i$ and a contact of second order at $x=1$, hence $k-1$ maxima. Accordingly, for only one position of a_i will y_2 touch y_1 in $k-1$ distinct points. These are the desired unique values for a_i, $(i=1, 2, 3, \cdots, k-1)$. Therefore only one such locus (1) can exist.

Two concentric circles $x^2+y^2 \pm r^2z^2=0$ polarize the cusps into the inflexional tangents (asymptotes perpendicular to cuspidal tangents). By the uniqueness of (1) these circles must autopolarize the entire locus. Since only *even* powers of x, y appear in the rectangular Cartesian form of (1), the locus is also autopolar by the rectangular hyperbolas $x^2-y^2 \pm r^2z^2=0$. By symmetry there must be $4k+2$ such hyperbolas, or $4k+4$ autopolarizing conics in all.

Non-degeneracy of the locus is observed from the following considerations. The locus has $2k+1$ biflecnodes at infinity, hence no finite inflexions; the nodes are all on cuspidal tangents, or on the bisectors of adjacent cuspidal tangents. For example, on the cuspidal tangent $\theta=0$ there are $k-1$ nodes. They can be produced only by the intersections of arms of cusps on either side of this ray, that is, by half the arms of $2(k-1)$ cusps. These arms must be asymptotic, joining with $2(k-1)$ other cuspidal arms (since no finite inflexions are permitted). Hence the second arms of the cusps adjacent to that at (0, 1) must have a vertical asymptote. Symmetry thus permits the curve to be sketched approximately by drawing secant lines through alternate points of $2(2k+1)$ points equally spaced about the unit circle, omitting the chords within the circle. Accordingly the locus is bipartite, each branch consisting of a set of alternate cusps with their infinitely long arms. If each branch is a proper curve, it must be also self-dual and of order $n/2$, (where $n=4k+2$), and possess $n/2$ cusps, and consequently $n(n-10)/8$ nodes. But one branch approximates $n/2$ secant lines through $n/2$ *consecutive* points equally spaced about the unit circle, and can then have only $n(n-6)/8$ nodes ($\neq n(n-10)/8$, as required). Therefore the locus is non-degenerate and consists of two congruent branches. These branches are polar reciprocal by $2k+2$ of the polarities, and individually autopolar by the other $2k+2$ polarities. Moreover the locus is invariant under $4k$ correlations in addition to the $4k+4$ polarities and $2(4k+2)$ collineations.

The equations and polarities are next exhibited for the two curves of lowest degree of this set.



T: Bipartite 10-ic autopolar by 10 rectangular hyperbolas and 2 circles.

Q, Q': Two of the 5 sets of conjugate hyperbolas which autopolarize the locus.

Q autopolarizes the two branches individually and has 4 real contacts with T.

Q' interchanges the two branches by the polarity.

CASE I.
$$k = 1, \qquad (n = 6).$$

Equation: $\rho^6 \sin^2 3\theta - (\rho^2-1)^3 = 0$, or $y^2(3x^2-y^2)^2-(x^2+y^2-z^2)^3 = 0$. Cusps: $(\pm 1, 0, 1)$, $(\pm 3^{1/2}, \pm 1, 2)$. Biflecnodes: $(0, 1, 0)$, $(2, \pm 3^{1/2}, 0)$. The eight polarities are readily noted among the twelve correlations:

$$\begin{array}{ccccc} u & : & v & : & w \\ \pm 3^{1/2}x & : & \pm 3^{1/2}y & : & z \\ 3^{1/2}x \pm 3y & : & 3x \pm 3^{1/2}y & : & \pm 2z \end{array}$$

CASE II.
$$k = 2, \qquad (n = 10).$$

Equation:
$$\rho^{10} \sin^2 5\theta - (\rho^2-1)^3(\rho^2-\alpha^2)^2 = 0,$$
$$\alpha^2 = 5[10(1/10)^{1/3}+3+2\cdot 10^{1/3}]/9,$$
or
$$y^2(5x^4 - 10x^2y^2 + y^4)^2 - (x^2+y^2-z^2)^3(x^2+y^2-\alpha^2z^2)^2 = 0, \qquad (\alpha = 2.576 \cdots).$$

Twelve polarities are readily noted among the twenty correlations.

$$\begin{array}{ccccc} u & : & v & : & w \\ 5\left(x \cos \dfrac{k\pi}{10} - y \sin \dfrac{k\pi}{10}\right) & : & \pm 5\left(x \sin \dfrac{k\pi}{10} + y \cos \dfrac{k\pi}{10}\right) & : & (2\alpha^2+3)^{1/2}z, \end{array}$$
$$(k = 0, 1, \cdots, 9).$$

SUMMARY. A completely symmetric, elliptic curve exists of order $4k+2$, having $4k+2$ cusps and $(2k+1)(4k-3)$ nodes, all distinct. The locus is invariant under $2(4k+2)$ correlations of which $4k+4$ are polarities. The locus is very approximately realized by drawing secant lines through the alternate points of $4k+2$ points equally spaced about the unit circle, omiting the chords. The locus is bipartite. Its $(2k+1)(4k-3)$ bitangents are very nearly the secants joining all pairs of the points of the circle which are separated by at least two of the points and which are not the ends of the same diameter.

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