A CYCLIC INVOLUTION OF ORDER SEVEN

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1. Introduction. In an earlier paper,† the writer discussed a cubic surface in ordinary three way space containing an involution of order five, \( I_5 \). This paper concerns itself with a different cubic surface which contains a cyclic involution, \( I_7 \).

2. Discussion of \( I_7 \) Belonging to \( F_2 \) in \( S_3 \). Consider the surface

\[ P_3(x_1, x_2, x_3, x_4) = ax^2_1 x_3 + bx^2_2 x_1 + cx_1 x_2 x_4 = 0 \]

in \( S_3 \), invariant under the cyclic collineation \( T \) of order seven

\[ x'_1 : x'_2 : x'_3 : x'_4 = x_1 : e x_2 : e^2 x_3 : e^3 x_4, \quad (e^7 = 1). \]

There are four invariant points, \( P_1 = (1, 0, 0, 0) \), \( P_2 = (0, 1, 0, 0) \), \( P_3 = (0, 0, 1, 0) \), and \( P_4 = (0, 0, 0, 1) \). Each lies on the surface \( F \), and since these are the only possible invariant points, the surface \( F \) has only four points of coincidence. It will be noticed, however, that only \( P_2 \) and \( P_3 \) are simple points of \( F \). Hence this paper will not be interested in the two double invariant points, \( P_1 \) and \( P_4 \).

Consider a curve \( C \), not transformed into itself by \( T \), and passing through \( P_2 \). Take the plane \( x_3 + \lambda x_4 = 0 \) of the pencil passing through \( P_2 \) and \( P_1 \), tangent to \( C \).

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$x_3 + \epsilon x_4 = 0$ by $T$ and hence is non-invariant. The curve cut out on $F$ by $x_3 + \lambda x_4 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Hence the two curves do not touch each other at $P_2$. Since $C$ was a variable curve through $P_2$ satisfying the non-invariant property, it follows that $P_2$ is a non-perfect coincidence point. A similar argument shows that $P_1$, $P_3$, and $P_4$ are also non-perfect coincidence points. The following theorem has been proved.

**Theorem 1.** The $I_7$ belonging to $F_3$ in $S_3$ has four non-perfect points of coincidence.

Consider the complete system of curves $|A|$ cut out on $F$ by all surfaces of order seven. Its dimension is 84, its genus is 64, and the number of variable intersections of two members of the system is 147. A curve $A$ of this system is not in general transformed into itself by $T$. There are, however, seven partial systems $|A_i|$ in $|A|$ which are transformed into themselves. By use of $|A_i|$ we find

\[ a_1 x_1^7 + a_2 x_2^7 + a_3 x_3^7 + a_4 x_1^7 + a_5 x_1^4 x_2 x_4^2 + a_6 x_1^4 x_2^2 x_4 + a_7 x_1^6 x_2^2 x_4 + a_8 x_1^6 x_2 x_4^2 + a_9 x_1^6 x_2 x_3 x_4 + a_{10} x_1^6 x_2 x_3^2 x_4 + a_{11} x_1^6 x_2 x_3 x_4^2 + a_{12} x_1^6 x_2 x_3^2 x_4^2 + a_{13} x_1^6 x_2 x_3 x_4^3 + a_{14} x_1^6 x_2 x_3^2 x_4^3 + a_{15} x_1^6 x_2 x_3 x_4^4 + a_{16} x_1^6 x_2 x_3^2 x_4^4 + a_{17} x_1^6 x_2 x_3 x_4^5 + a_{18} x_1^6 x_2 x_3^2 x_4^5 = 0.\]

We refer the curves $A_1$ projectively to the hyperplanes of a linear space of seventeen dimensions. We obtain a surface $\phi$, of order 21, with hyperplane sections of genus 10, as the image of $I_7$. The equations of the transformation for mapping $I_7$ upon $\phi$ in $S_{17}$ are

\[ \rho X_1 = x_1^7, \quad \rho X_2 = x_2^7, \quad \rho X_3 = x_3^7, \quad \rho X_4 = x_4^7, \quad \rho X_5 = x_5 x_2 x_4^2, \quad \rho X_6 = x_5 x_3 x_4^2, \quad \rho X_7 = x_5 x_2 x_3 x_4, \quad \rho X_8 = x_5 x_3 x_4 x_3, \quad \rho X_9 = x_5 x_4 x_3 x_4, \quad \rho X_{10} = x_5 x_4 x_3 x_4, \quad \rho X_{11} = x_5 x_4 x_3 x_4, \quad \rho X_{12} = x_5 x_4 x_3 x_4, \quad \rho X_{13} = x_5 x_4 x_3 x_4. \]

By eliminating $\rho, x_1, x_2, x_3, x_4$ from these eighteen equations and from $F_3(x_1 x_3 x_5 x_4) = 0$, we get as the fifteen equations defining the surface:

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\[ \begin{vmatrix} X_1 & X_5 & X_8 & X_7 & X_6 \\ X_5 & X_{13} & X_{17} & X_{18} & X_{12} \end{vmatrix} = 0, \quad \begin{vmatrix} X_2 & X_{10} & X_{11} & X_{18} & X_{15} \\ X_{10} & X_5 & X_6 & X_9 & X_7 \end{vmatrix} = 0, \]
\[ \begin{vmatrix} X_3 & X_{14} & X_{17} & X_{16} \\ X_{14} & X_9 & X_{13} & X_{12} \end{vmatrix} = 0, \quad \begin{vmatrix} X_4 & X_{13} & X_{12} \\ X_9 & X_7 & X_8 \end{vmatrix} = 0, \quad \begin{vmatrix} X_{18} & X_{12} \\ X_{17} & X_{14} \end{vmatrix} = 0, \]

and \( aX_{17} + bX_{14} + cX_{12} = 0 \). Designate by \( P'_2 \) the branch point of \( \phi \) corresponding to the point \( P_2 \) on \( F \). The coordinates of \( P'_2 \) are all zero except \( X_2 \).

The curves \( A_1 \) on \( F \) pass through \( P_2 \) if \( a_2 = 0 \). The tangent plane at \( P_3 \) to \( F \) is \( x_3 = 0 \). Now, the system of seventh-degree surfaces passing through \( P_2 \) cuts \( x_3 = 0 \) in the curves \( x_3 = 0 \), and

\[ a_1x_1^2 + a_4x_1^4 + a_6x_1^4x_2x_3^2 + a_{10}x_1^7x_4 + a_{13}x_1x_2^2x_4 = 0. \]

For general values of the constants this is a seventh-degree curve with a triple point at \( P_2 \), two branches being tangent to the line \( x_1 = x_3 = 0 \) and one to the line \( x_3 = x_4 = 0 \). When \( a_5 = a_{10} = a_{13} = 0 \), the plane seventh-degree curve breaks up into seven lines through \( P_2 \). These are all distinct except when either \( a_1 = 0 \) or \( a_4 = 0 \), when they coincide with \( x_3 = x_4 = 0 \) or \( x_1 = x_3 = 0 \), respectively. Since \( P_2 \) is non-perfect, the \( |A_1| \) through \( P_2 \) must have seven distinct branches unless each branch touches one of the two invariant directions. In the plane \( x_3 = 0 \), the involution \( I_7 \) is generated by the homography \( T_2 \), which is

\[ x'_1 : x'_2 : x'_4 = x_1 : \varepsilon x_2 : \varepsilon^3 x_4. \]

By use of the plane quadratic transformation \( X : y_1 : y_2 : y_4 =\end{vmatrix} w_1w_4 : w_2^2 : w_3w_2 \) and its inverse \( X^{-1} : w_1 : w_3^2 : w_4 = y_1^2 : y_2y_4 : y_1y_2 \) as well as the transformation \( Y : y_1 : y_2 : y_4 = w_1w_4 : w_2^2 : w_3w_2 \) and its inverse \( Y^{-1} : w_1 : w_3^2 : w_4 = y_2y_4 : y_1y_2 : y_1^2 \), we can investigate the character of the adjacent invariant points along the two invariant directions at \( P_2 \). By the application of \( XT_2X^{-1} = T'_2 \),

\[ (w_1, w_3, w_4) \underset{X^{-1}}{\sim} (y_1^2, y_2y_4, y_1y_2) \underset{T_2}{\sim} (e^5y_1^2, e^4y_2y_4, e^3y_1y_2), \]

or

\[ (e^5y_1^2, e^4y_2y_4, y_1y_2) \underset{T'_2}{\sim} (e^6w_1, e^3w_2, w_4). \]

Thus the new transformation \( T'_2 \) is \( x'_1 : x'_2 : x'_4 = e^6x_1 : e^3x_2 : x_4 \). The invariant point adjacent to \( P_2 \) along the line \( x_1 = x_3 = 0 \) is still a non-perfect coincidence point. Using on the next point
YT_2 Y^{-1} \equiv T_2''$, we find $(w_1, w_2, w_4) T_2'' (w_1, \epsilon w_2, w_4)$. This point is a perfect point of coincidence. This means that $T_2''$ is $x'_1 : x'_2 : x'_4 = x_1 : \epsilon x_2 : x_4$ and hence it is the collineation representing a perfect point for $(0, 1, 0)$. Thus, by $XT_2 X^{-1}$ and $YT_2 Y^{-1}$, one finds a perfect point along $x_1 = x_3 = 0$ in the neighborhood of the second order of $P_2$. Therefore the following is true.

**Theorem 2.** Along the invariant direction $x_1 = x_3 = 0$, the invariant point $P_2$ has an imperfect point in the first-order neighborhood and a perfect one in the second-order neighborhood.

Next, investigate the characteristics of the adjacent point to $P_2$ along the invariant direction $x_3 = x_4 = 0$. By use of $YT_2 Y^{-1} \equiv T_2''$, we get

$$(w_1, w_2, w_4) \overset{Y}{\sim} (x_2 x_4, x_1 x_2, x_4^2) T_1 \overset{T}{\sim} (\epsilon^4 x_2 x_4, \epsilon x_1 x_2, x_4^2) \overset{Y}{\sim} (\epsilon^4 w_1, \epsilon w_2, w_4).$$

Hence $T_2''$ is $x'_1 : x'_2 : x'_4 = \epsilon^4 x_1 : \epsilon x_2 : x_4$ and this indicates an imperfect point adjacent to $P_2$ along $x_3 = x_4 = 0$. Apply $YT_2 Y^{-1} \equiv T_2'''$ and find $(w_1, w_2, w_4) \sim (w_1, \epsilon^4 w_2, w_4)$. Since $T_2'''$ becomes $x'_1 : x'_2 : x'_4 = x_1 : \epsilon^4 x_2 : x_4$, we are assured of a perfect point in the second-order neighborhood of $P_2$ along this invariant direction. Hence the theorem follows.

**Theorem 3.** Along the invariant direction $x_4 = x_3 = 0$, the invariant imperfect point $P_2$ has an imperfect point in the first-order neighborhood and a perfect one in the second-order neighborhood.

The following theorem is now self-evident.

**Theorem 4.** The imperfect point $P_2$ on $F_3$ has no perfect points in the neighborhood of the first order but precisely two perfect ones in the neighborhood of the second order.

The tangent plane to $F$ at $P_3 = (0, 0, 1, 0)$ is $x_1 = 0$. The homography $T_3$ in $x_1 = 0$ is $x'_1 : x'_2 : x'_4 = x_2 : \epsilon x_3 : \epsilon^2 x_4$. To investigate the adjacent points to $P_3$ along the two invariant directions $x_1 = x_2 = 0$ and $x_1 = x_4 = 0$, one needs the following two quadratic transformations and their inverses:

\[
Y_1: \quad y_2 : y_3 : y_4 = w_3 w_4 : w_2^2 : w_2 w_4, \\
Y_1^{-1}: \quad w_2 : w_3 : w_4 = y_3 y_4 : y_2 y_3 : y_2^2, \\
X_1: \quad y_2 : y_3 : y_4 = w_2 w_4 : w_2^2 : w_2 w_3, \\
X_1^{-1}: \quad w_2 : w_3 : w_4 = y_2^2 : y_3 y_4 : y_2 y_3.
\]
Apply $X_1T_3X_1^{-1}=T_3'$ along $x_1=x_4=0$ adjacent to $P_3$. Then we have

$$(w_2, w_3, w_4)^X \sim (y_2^2, y_3y_4, y_2y_3) \quad T_3 \quad (e^2y_2^2, e^4y_3y_4, ey_2y_3)$$

Since $T_3'$ is $x_1':x_3' = e^2x_3':e^2x_4$, we have an imperfect point. By using $Y_1T_3Y_1^{-1}=T_3''$, we get

$$(w_2, w_3, w_4)^{T_3''} \sim (e^2w_2, e^3w_3, e^2w_4).$$

Hence we have a perfect point.

**Theorem 5.** Along the invariant direction $x_1=x_2=0$, the invariant imperfect point $P_3$ has an imperfect adjacent point and a perfect one in the neighborhood of the second order.

Now consider the possibilities along $x_1=x_4=0$, the other invariant direction. Apply $Y_1T_3Y_1^{-1}=T_3^{(t)}$ and get

$$(w_2, w_3, w_4)^{T_3^{(t)}} \sim (e^3w_2, e^2w_3, e^2w_4),$$

which signifies an imperfect point. Applying $Y_1T_3^{(t)}Y_1^{-1}=T_3^{(tt)}$, we get $(w_2, w_3, w_4)^{T_3^{(tt)}} \sim (e^3w_2, ew_3, e^2w_4)$, another imperfect point. By use of $X_1T_3^{(tt)}X_1^{-1}=T_3^{(ttt)}$, we get $(w_2, w_3, w_4) \sim (e^3w_2, ew_3, e^2w_4)$, which indicates that $T_3^{(ttt)}$ is $x_1':x_3'.x_4'=e^3x_3':e^2x_4$. Hence we have found a perfect point, and the theorem follows.

**Theorem 6.** Along the invariant direction $x_1=x_4=0$, the invariant imperfect point $P_3$ has no perfect points in the first- and second-order neighborhoods but does have one in the third-order neighborhood.

3. **Sections by Sextics.** Consider the complete system of curves $|B|$ cut out on $F$ by all surfaces of order six. Its dimension is 63, its genus is 46, and the number of variable intersections of two members of the system is 108. A curve $B$ of this system is not in general transformed into itself. There are, however, seven partial systems $|B_i|$ in $|B|$ which are transformed into themselves. By use of $|B_1|$ we find

$$b_1x_1x_4^5 + b_2x_2x_3x_4^4 + b_3x_3^3x_4^3 + b_4x_1^2x_3^2x_4^2 + b_5x_1x_3^2x_4 + b_6x_3^4x_4 + b_7x_1x_3^2x_4^4 + b_8x_1x_3^2x_4^4 + b_9x_3^4x_4 + b_{10}x_1x_3^2x_4^3 + b_{11}x_2x_4^2 + b_{12}x_4^5x_2 = 0.$$  

If we refer the curves $B_1$ projectively to the hyperplanes of a
linear space of eleven dimensions, we obtain a surface \( \phi \). The equations of transformation for mapping \( I_7 \) upon \( \phi \) in \( S_{11} \) are

\[
\begin{align*}
\rho X_1 &= x_1 x_4^5, & \rho X_5 &= x_3 x_4^5, & \rho X_9 &= x_1^2 x_3^7, \\
\rho X_2 &= x_2 x_3 x_4^4, & \rho X_6 &= x_2^5 x_4, & \rho X_{10} &= x_1 x_2^5 x_4^7, \\
\rho X_3 &= x_3^5 x_4, & \rho X_7 &= x_1 x_2^3 x_3 x_4, & \rho X_{11} &= x_2^4 x_3^5, \\
\rho X_4 &= x_1^3 x_2^5 x_4^2, & \rho X_8 &= x_2^3 x_3^3 x_4, & \rho X_{12} &= x_1^4 x_2.
\end{align*}
\]

By eliminating \( \rho \), \( x_1 \), \( x_2 \), \( x_3 \), \( x_4 \) from these twelve equations and from \( F_5(x_1, x_2, x_3, x_4) = 0 \), we get as the nine equations defining the surface

\[
\begin{bmatrix}
X_1 & X_4 & X_7 \\
X_2 & X_7 & X_{11}
\end{bmatrix} = 0, \quad \begin{bmatrix}
X_4 & X_7 & X_6 \\
X_8 & X_{10} & X_{11}
\end{bmatrix} = 0, \\
\begin{bmatrix}
X_7 & X_{10} & X_6 \\
X_8 & X_9 & X_7
\end{bmatrix} = 0, \quad \begin{bmatrix}
X_5 & X_2 & X_3 \\
X_{12} & X_4 & X_8
\end{bmatrix} = 0,
\]

and \( aX_7 + bX_8 + cX_4 = 0 \).

All the curves \( B_1 \) pass through the invariant points \( P_1, P_2, P_3, P_4 \). Consider point \( P_2 \). Its tangent plane is \( x_3 = 0 \). It cuts the sextic surfaces in the curves

\[
x_3 = 0, \quad b_1 x_1 x_4^3 + b_4 x_1^5 x_4^2 + b_6 x_1^5 x_4 + b_{12} x_1 x_4^5 x_2 = 0.
\]

This curve passes simply through \( P_2 \) along the invariant direction \( x_3 = x_4 = 0 \). If \( b_6 = 0 \), the curve degenerates into a line and a quintic with a triple point formed by a simple branch passing through a cusp. The line is the simple tangent at the triple point, while the cuspidal tangent cuts the curve again at an undulation. If \( b_6 = b_4 = 0 \), the curve degenerates into a quintic with a five-fold point (having five coincident tangents) and a line, the tangent at the five-fold point. If \( b_6 = b_4 = b_{12} = 0 \), \( P_2 \) is a six-fold point. The curve breaks up into six lines, one of which is \( x_1 = x_3 = 0 \), while the other five are \( x_3 = x_4 = 0 \) counted five times. Thus the system of \( B_1 \) curves passes through \( P_2 \) along invariant directions.

The tangent plane at \( P_3 \) is \( x_1 = 0 \). It cuts the sextic surfaces in the curves \( x_1 = 0, b_2 x_2 x_3 x_4 + b_5 x_3^3 x_4^2 + b_6 x_2 x_4 + b_{12} x_2^3 x_3^2 = 0 \). \( P_2 \) is a special triple point on these curves having three coincident tangents. The curves touch at \( P_3 \) along the invariant direction \( x_1 = x_4 = 0 \). If \( b_3 = 0 \), then the curve passes through \( P_3 \) four times and is tangent to \( x_1 = x_2 = 0 \) four times. When \( b_4 = b_{11} = 0 \), the
curve is five fold at \( P_3 \), passing through along the invariant direction \( x_1 = x_2 = 0 \) once and along \( x_1 = x_4 = 0 \) four times. When \( b_3 = b_{11} = b_2 = 0 \), the curve breaks up into six lines through \( P_3 \), namely, \( x_1 = x_4 = 0 \) counted simply and \( x_1 = x_2 = 0 \) counted five times. The following theorem may be stated.

**Theorem 7.** The \( |B_1| \) curves pass through the imperfect points only along the invariant directions.

4. **Sections by Quintics.** The complete system of curves \( |C| \) cut out on \( F \) by all surfaces of order five has dimension 45, genus 31, and the number of variable intersections of two members of the system is 75. There are seven partial systems \( |C_1| \) in \( |C| \) which are transformed into themselves. By the use of \( |C_1| \) we find

\[
c_1 x_1^2 x_4 + c_2 x_1 x_3 x_3 x_4 + c_3 x_2^2 x_4 + c_4 x_1 x_3 x_4 + c_5 x_2^2 x_3 x_4 + c_6 x_2 x_3 + c_7 x_1 x_3 + c_8 x_7 x_8^2 = 0.
\]

By referring the curves \( C_1 \) projectively to the hyperplanes of a linear space of seven dimensions, we obtain a surface \( \phi \). The equations of transformation for mapping \( I_7 \) upon \( \phi \) in \( S_7 \) are

\[
\rho X_1 = x_1^2 x_3, \quad \rho X_3 = x_3^2 x_4, \quad \rho X_5 = x_2^2 x_3 x_4, \quad \rho X_7 = x_1 x_3 x_4, \\
\rho X_2 = x_1 x_2 x_3 x_4, \quad \rho X_4 = x_1 x_3 x_4, \quad \rho X_6 = x_2 x_3^4, \quad \rho X_8 = x_1 x_2^2.
\]

Eliminate \( \rho \), \( x_1 \), \( x_2 \), \( x_3 \), \( x_4 \) from these eight equations and from \( F_5(x_1, x_2, x_3, x_4) = 0 \). The five equations defining the surface \( \phi \) are

\[
\begin{bmatrix}
X_1 & X_2 & X_4 \\
X_2 & X_5 & X_6
\end{bmatrix} = 0, \quad \begin{bmatrix}
X_3 & X_5 \\
X_5 & X_6
\end{bmatrix} = 0, \quad \begin{bmatrix}
X_2 & X_7 \\
X_3 & X_8
\end{bmatrix} = 0,
\]

and

\[aX_5 + bX_4 + cX_2 = 0.\]

All the curves \( C_1 \) pass through the invariant points \( P_1, P_2, P_3, P_4 \). Its tangent plane at \( P_2 \) is \( x_3 = 0 \). It cuts the quintic surfaces in the curves \( x_3 = 0, c_1 x_1^2 x_3 + c_3 x_2^2 + c_5 x_1^2 x_2^2 + c_8 x_1^3 x_2^2 = 0 \). This curve has a double point at \( P_2 \), both branches being tangent to the invariant direction \( x_3 = x_4 = 0 \). When \( c_3 = 0 \), the curve degenerates into a cuspidal cubic and a repeated line (the flex tangent, which is \( x_1 = x_3 = 0 \)). If \( c_3 = c_5 = 0 \), the curve degenerates into five straight lines through \( P_2 \). They are \( x_1 = x_3 = 0 \) counted twice and
\( x_3 = x_4 = 0 \) counted three times. Hence these curves pass through \( P_2 \) along invariant directions.

The tangent plane to \( F_3 \) at \( P_3 \) is \( x_1 = 0 \). It cuts the quintic surfaces in the curves \( x_1 = 0, c_3 x_2^3 x_4^2 + c_5 x_2^2 x_3^2 x_4 + c_6 x_2 x_3^4 = 0 \). This curve is simple at \( P_3 \), passing through along the invariant direction \( x_1 = x_2 = 0 \). When \( c_6 = 0 \), the curve degenerates into a conic and three lines, one line \((x_1 = x_4 = 0)\) a simple tangent and the other \((x_1 = x_2 = 0)\) a repeated tangent. If \( c_6 = c_5 = 0 \), then the curve breaks up into the line \( x_1 = x_2 = 0 \) counted three times and the line \( x_1 = x_4 = 0 \) counted twice.

**Theorem 8.** The \(|C_1|\) curves pass through imperfect points only along invariant directions.

5. **Sections by Quartics.** The dimension of the complete system of curves \(|D|\) cut out on \( F \) by all surfaces of order four is 30, its genus is 19, and the number of variable intersections of two members of the system is 48. By use of \(|D|\) we find

\[
d_1 x_2 x_3^8 + d_2 x_3^8 x_4^2 + d_3 x_2^3 x_4 + d_4 x_1 x_3^3 + d_5 x_2 x_2 x_3 = 0.
\]

We refer the curves \( D_1 \) projectively to the hyperplanes of a linear space of four dimensions, and obtain a surface \( \phi \). The equations of transformation for mapping \( I_7 \) upon \( \phi \) in \( S_4 \) are

\[
\rho X_1 = x_2 x_1^8, \rho X_2 = x_3^8 x_4, \rho X_3 = x_1^8 x_4, \rho X_4 = x_1 x_3^8, \rho X_5 = x_1^2 x_2 x_3.
\]

The two equations defining the surface are

\[
X_1 X_2^2 = X_2 X_3 X_4 \quad \text{and} \quad a X_1 X_5 + b X_2 X_3 + c X_1 X_3 = 0.
\]

All the curves \( D_1 \) pass through the invariant points \( P_1, P_2, P_3, \) and \( P_4 \). Consider the point \( P_2 \). Its tangent plane is \( x_3 = 0 \). It cuts the quartic surfaces in the curves

\[
x_3 = 0, \quad d_1 x_2 x_4^3 + d_3 x_3^3 x_4 + d_4 x_1 x_3^3 = 0.
\]

This curve passes simply through \( P_2 \) along the \( x_1 = x_3 = 0 \) direction. When \( d_4 = 0 \), the curve degenerates into a cuspidal cubic and the cusp tangent \((x_3=x_4=0)\). When \( d_4 = d_1 = 0 \), the curve breaks up into four lines through \( P_2 \). They are \( x_1 = x_3 = 0 \) counted three times and \( x_3 = x_4 = 0 \) counted once.

The tangent plane at \( P_3 \) cuts the quartic surfaces in the curves

\[
x_1 = 0, \quad d_1 x_2 x_4^3 + d_3 x_3^3 x_4^2 = 0.
\]

The quartic curves degenerate into...
conics and a repeated (tangent) line. When \( d_2 = 0 \), they break up into \( x_1 = x_4 = 0 \) counted three times and \( x_1 = x_2 = 0 \) counted once.

Theorem 9. The \( |D_1| \) curves pass through imperfect points only along invariant directions.

6. Sections by Cubics. Investigate the complete system of curves \( |E| \) cut out on \( F \) by all surfaces of order three. Its dimension is 19, genus is 10, and the number of variable intersections of the two members of the system is 27. The use of \( |E_1| \) gives
\[
e_1 x_2^2 x_3 + e_2 x_1 x_3^2 + e_3 x_1 x_2 x_4 = 0.
\]
The equations of transformation for referring the curves \( E_1 \) projectively to the lines of a plane are
\[
\rho X_1 = x_2^2 x_3, \quad \rho X_2 = x_1 x_3^2, \quad \rho X_3 = x_1 x_2 x_4.
\]
A curve \( aX_1 + bX_2 + cX_3 = 0 \) is obtained, instead of a surface. All the curves \( E_1 \) pass through the invariant points \( P_1, P_2, P_3, \) and \( P_4 \). The tangent plane to \( F_3 \) at \( P_2 \) is \( x_3 = 0 \). This intersects \( E_1 \) surfaces in \( x_3 = 0, x_1 x_2 x_4 = 0 \). Hence, the cubic curve becomes three straight lines, two of which pass through \( P_2 \), namely, \( x_1 = x_3 = 0 \) and \( x_3 = x_4 = 0 \). At \( P_3 \) the tangent plane is \( x_1 = 0 \). It cuts the \( E_1 \) surfaces in \( x_1 = 0, x_3 x_4 = 0 \). This degenerate cubic curve also has a double point at \( P_3 \). The branches are \( x_1 = x_2 = 0 \) counted twice. Hence the following theorem is proved.

Theorem 10. The system of invariant curves cut out upon \( F \) by surfaces of degree lower than seven all pass through the coincidence points along the invariant directions. The number of branches through each point is less than seven.

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