A CYCLIC INVOLUTION OF ORDER SEVEN

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1. *Introduction.* In an earlier paper,† the writer discussed a cubic surface in ordinary three way space containing an involution of order five, I_5. This paper concerns itself with a different cubic surface which contains a cyclic involution, I_7.

2. *Discussion of I_7 Belonging to F_3 in S_3.* Consider the surface

$$F_3(x_1,x_2,x_3,x_4) \equiv ax_2^2 x_3 + bx_3^2 x_1 + cx_1x_2x_4 = 0$$

in S_3, invariant under the cyclic collineation T of order seven

$$x_1' : x_2' : x_3' : x_4' = x_1 : \epsilon x_2 : \epsilon^2 x_3 : \epsilon^3 x_4, \qquad (\epsilon^7 = 1).$$

There are four invariant points, $P_1 \equiv (1, 0, 0, 0)$, $P_2 \equiv (0, 1, 0, 0)$, $P_3 \equiv (0, 0, 1, 0)$, and $P_4 \equiv (0, 0, 0, 1)$. Each lies on the surface F, and since these are the only possible invariant points, the surface F has only four points of coincidence. It will be noticed, however, that only P_2 and P_3 are simple points of F. Hence this paper will not be interested in the two double invariant points, P_1 and P_4.

Consider a curve C, not transformed into itself by T, and passing through P_2. Take the plane $x_3+\lambda x_4=0$ of the pencil passing through P_2 and P_1, tangent to C. This plane is transformed into

P28 and its equivalent P6 are regarded as part of the "formal" theory; but both may be omitted, if preferred, without prejudice to the other postulates.)

What is perhaps the most obvious example of a "formal Principia system with equality" is the system (K, C, $+$, $'$, $=$) obtained from Example 0.4 by changing the word "correct" to "truistic." The resulting example satisfies all the Postulates P1–P6, P8–P11, but fails on P7 (since there are verdicts a such that neither a nor a' is a "truistic" verdict).

Thus the distinction between an "*informal* Principia system with equality" and a "*formal* Principia system with equality" depends on the inclusion or rejection of Postulate P7.

It is important to observe, however, that another, equally good, example of a "formal Principia system with equality" is the system obtained from Example 0.5 by changing the word "incorrect" to "absurd." The mathematical postulates by themselves give no precedence to the "truistic-or" interpretation over the "absurd-and" interpretation.

† W. R. Hutcherson, *Maps of certain cyclic involutions on two-dimensional carriers*, this Bulletin, vol. 37 (1931), pp. 759–765.

$x_3+\epsilon\lambda x_4=0$ by T and hence is non-invariant. The curve cut out on F by $x_3+\lambda x_4=0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Hence the two curves do not touch each other at P_2. Since C was a variable curve through P_2 satisfying the non-invariant property, it follows that P_2 is a non-perfect coincidence point. A similar argument shows that P_1, P_3, and P_4 are also non-perfect coincidence points. The following theorem has been proved.

THEOREM 1. *The I_7 belonging to F_3 in S_3 has four non-perfect points of coincidence.*

Consider the complete system of curves $|A|$ cut out on F by all surfaces of order seven. Its dimension is 84, its genus is 64, and the number of variable intersections of two members of the system is 147. A curve A of this system is not in general transformed into itself by T. There are, however, seven partial systems $|A_i|$ in $|A|$ which are transformed into themselves. By use of $|A_1|$ we find

$$\begin{aligned} &a_1x_1^7+a_2x_2^7+a_3x_3^7+a_4x_4^7+a_5x_1^4x_2x_4^2+a_6x_1^4x_3^2x_4\\ &+a_7x_1^3x_2^2x_3x_4+a_8x_1^3x_2x_3^3+a_9x_1^2x_3x_4^4+a_{10}x_1^2x_2^4x_4\\ &+a_{11}x_1^2x_2^3x_3^2+a_{12}x_1x_2x_3^2x_4^3+a_{13}x_1x_2^2x_4^4+a_{14}x_1x_3^4x_4^2\\ &+a_{15}x_1x_2^5x_3+a_{16}x_2x_3^5x_4+a_{17}x_2^2x_3^3x_4^2+a_{18}x_2^3x_3x_4^3=0. \end{aligned}$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of seventeen dimensions. We obtain a surface ϕ, of order 21, with hyperplane sections of genus 10, as the image of I_7. The equations of the transformation for mapping I_7 upon ϕ in S_{17} are

$$\begin{array}{lll} \rho X_1=x_1^7, & \rho X_7=x_1^3x_2^2x_3x_4, & \rho X_{13}=x_1x_2^2x_4^4,\\ \rho X_2=x_2^7, & \rho X_8=x_1^3x_2x_3^3, & \rho X_{14}=x_1x_3^4x_4^2,\\ \rho X_3=x_3^7, & \rho X_9=x_1^2x_3x_4^4, & \rho X_{15}=x_1x_2^5x_3,\\ \rho X_4=x_4^7, & \rho X_{10}=x_1^2x_2^4x_4, & \rho X_{16}=x_2x_3^5x_4,\\ \rho X_5=x_1^4x_2x_4^2, & \rho X_{11}=x_1^2x_2^3x_3^2, & \rho X_{17}=x_2^2x_3^3x_4^2,\\ \rho X_6=x_1^4x_3^2x_4, & \rho X_{12}=x_1x_2x_3^2x_4^3, & \rho X_{18}=x_2^3x_3x_4^3. \end{array}$$

By eliminating ρ, x_1, x_2, x_3, x_4 from these eighteen equations and from $F_3(x_1x_2x_3x_4)=0$, we get as the fifteen equations defining the surface:

$$\left\|\begin{matrix} X_1 & X_5 & X_8 & X_7 & X_6 \\ X_5 & X_{13} & X_{17} & X_{18} & X_{12} \end{matrix}\right\| = 0, \quad \left\|\begin{matrix} X_2 & X_{10} & X_{11} & X_{18} & X_{15} \\ X_{10} & X_5 & X_6 & X_9 & X_7 \end{matrix}\right\| = 0,$$

$$\left\|\begin{matrix} X_3 & X_{14} & X_{17} & X_{16} \\ X_{14} & X_9 & X_{13} & X_{12} \end{matrix}\right\| = 0, \left\|\begin{matrix} X_4 & X_{13} & X_{12} \\ X_9 & X_7 & X_8 \end{matrix}\right\| = 0, \left\|\begin{matrix} X_{18} & X_{12} \\ X_{17} & X_{14} \end{matrix}\right\| = 0,$$

and $aX_{17}+bX_{14}+cX_{12}=0$. Designate by P_2' the branch point of ϕ corresponding to the point P_2 on F. The coordinates of P_2' are all zero except X_2.

The curves A_1 on F pass through P_2 if $a_2=0$. The tangent plane at P_2 to F is $x_3=0$. Now, the system of seventh-degree surfaces passing through P_2 cuts $x_3=0$ in the curves $x_3=0$, and

$$a_1x_1^7 + a_4x_4^7 + a_5x_1^4x_2x_4^2 + a_{10}x_1^2x_2^4x_4 + a_{13}x_1x_2^2x_4^4 = 0.$$

For general values of the constants this is a seventh-degree curve with a triple point at P_2, two branches being tangent to the line $x_1=x_3=0$ and one to the line $x_3=x_4=0$. When $a_5=a_{10}=a_{13}=0$, the plane seventh-degree curve breaks up into seven lines through P_2. These are all distinct except when either $a_1=0$ or $a_4=0$, when they coincide with $x_3=x_4=0$ or $x_1=x_3=0$, respectively. Since P_2 is non-perfect, the $|A_1|$ through P_2 must have seven distinct branches unless each branch touches one of the two invariant directions. In the plane $x_3=0$, the involution I_7 is generated by the homography T_2, which is

$$x_1' : x_2' : x_4' = x_1 : \epsilon x_2 : \epsilon^3 x_4.$$

By use of the plane quadratic transformation X: $y_1:y_2:y_4=w_1w_4:w_2^2:w_1w_2$ and its inverse X^{-1}: $w_1:w_2:w_4=y_4^2:y_2y_4:y_1y_2$ as well as the transformation Y: $y_1:y_2:y_4=w_2w_4:w_2^2:w_1w_4$ and its inverse Y^{-1}: $w_1:w_2:w_4=y_2y_4:y_1y_2:y_1^2$, we can investigate the character of the adjacent invariant points along the two invariant directions at P_2. By the application of $XT_2X^{-1}\equiv T_2'$,

$$(w_1,w_2,w_4) \overset{X^{-1}}{\sim} (y_4^2, y_2y_4, y_1y_2) \overset{T_2}{\sim} (\epsilon^6y_4^2, \epsilon^4y_2y_4, \epsilon y_1y_2),$$

or

$$(\epsilon^5y_4^2, \epsilon^3y_2y_4, y_1y_2) \overset{X}{\sim} (\epsilon^5w_1, \epsilon^3w_2, w_4).$$

Thus the new transformation T_2' is $x_1':x_2':x_4'=\epsilon^5x_1:\epsilon^3x_2:x_4$. The invariant point adjacent to P_2 along the line $x_1=x_3=0$ is still a non-perfect coincidence point. Using on the next point

$YT_2' Y^{-1} \equiv T_2''$, we find $(w_1, w_2, w_4) \overset{T_2'}{\sim} (w_1, \epsilon^5 w_2, w_4)$. This point is a perfect point of coincidence. This means that T_2'' is $x_1' : x_2' : x_4' = x_1 : \epsilon^5 x_2 : x_4$ and hence it is the collineation representing a perfect point for (0, 1, 0). Thus, by XT_2X^{-1} and $YT_2' Y^{-1}$, one finds a perfect point along $x_1 = x_3 = 0$ in the neighborhood of the second order of P_2. Therefore the following is true.

THEOREM 2. *Along the invariant direction* $x_1 = x_3 = 0$, *the invariant point* P_2 *has an imperfect point in the first-order neighborhood and a perfect one in the second-order neighborhood.*

Next, investigate the characteristics of the adjacent point to P_2 along the invariant direction $x_3 = x_4 = 0$. By use of $YT_2Y^{-1} \equiv T_2^{(\prime)}$ we get

$$(w_1, w_2, w_4) \overset{Y^{-1}}{\sim} (x_2x_4, x_1x_2, x_1^2) \overset{T_2}{\sim} (\epsilon^4 x_2x_4, \epsilon x_1x_2, x_1^2)$$

$$\overset{Y}{\sim} (\epsilon^4 w_1, \epsilon w_2, w_4).$$

Hence $T_2^{(\prime)}$ is $x_1' : x_2' : x_4' = \epsilon^4 x_1 : \epsilon x_2 : x_4$ and this indicates an imperfect point adjacent to P_2 along $x_3 = x_4 = 0$. Apply $Y\ T_2'\ Y^{-1} \equiv T_2^{(\prime\prime)}$ and find $(w_1, w_2, w_4) \sim (w_1, \epsilon^4 w_2, w_4)$. Since $T_2^{(\prime\prime)}$ becomes $x_1' : x_2' : x_4' = x_1 : \epsilon^4 x_2 : x_4$, we are assured of a perfect point in the second-order neighborhood of P_2 along this invariant direction. Hence the theorem follows.

THEOREM 3. *Along the invariant direction* $x_4 = x_3 = 0$, *the invariant imperfect point* P_2 *has an imperfect point in the first-order neighborhood and a perfect one in the second-order neighborhood.*

The following theorem is now self-evident.

THEOREM 4. *The imperfect point* P_2 *on* F_3 *has no perfect points in the neighborhood of the first order but precisely two perfect ones in the neighborhood of the second order.*

The tangent plane to F at $P_3 \equiv (0, 0, 1, 0)$ is $x_1 = 0$. The homography T_3 in $x_1 = 0$ is $x_2' : x_3' : x_4' = x_2 : \epsilon x_3 : \epsilon^2 x_4$. To investigate the adjacent points to P_3 along the two invariant directions $x_1 = x_2 = 0$ and $x_1 = x_4 = 0$, one needs the following two quadratic transformations and their inverses:

$$\begin{aligned} Y_1 &: \quad y_2 : y_3 : y_4 = w_3w_4 : w_3^2 : w_2w_4, \\ Y_1^{-1} &: \quad w_2 : w_3 : w_4 = y_3y_4 : y_2y_3 : y_2^2, \\ X_1 &: \quad y_2 : y_3 : y_4 = w_2w_4 : w_3^2 : w_2w_3, \\ X_1^{-1} &: \quad w_2 : w_3 : w_4 = y_4^2 : y_3y_4 : y_2y_3. \end{aligned}$$

Apply $X_1T_3X_1^{-1}\equiv T_3'$ along $x_1=x_4=0$ adjacent to P_3. Then we have

$$(w_2,\ w_3,\ w_4)\overset{X_1^{-1}}{\sim}(y_4^2,\ y_3y_4,\ y_2y_3)\overset{T_3}{\sim}(\epsilon^2y_4^2,\ \epsilon^3y_3y_4,\ \epsilon y_2y_3)$$

$$\overset{X_1}{\sim}(\epsilon w_2,\ \epsilon^2w_3,\ w_4).$$

Since T_3' is $x_2':x_3':x_4'=\epsilon x_2:\epsilon^2x_3:x_4$, we have an imperfect point. By using $Y_1T_3'\,Y_1^{-1}=T_3''$, we get

$$(w_2,\ w_3,\ w_4)\overset{T_3''}{\sim}(\epsilon^2w_2,\ \epsilon^3w_3,\ \epsilon^2w_4).$$

Hence we have a perfect point.

THEOREM 5. *Along the invariant direction $x_1=x_2=0$, the invariant imperfect point P_3 has an imperfect adjacent point and a perfect one in the neighborhood of the second order.*

Now consider the possibilities along $x_1=x_4=0$, the other invariant direction. Apply $Y_1T_3Y_1^{-1}\equiv T_3^{(\prime)}$ and get

$$(w_2,\ w_3,\ w_4)\sim(\epsilon^3w_2,\ \epsilon w_3,\ w_4),$$

which signifies an imperfect point. Applying $Y_1T_3^{(\prime)}Y_1^{-1}\equiv T_3^{(\prime\prime)}$, we get $(w_2,\ w_3,\ w_4)\sim(w_2,\ \epsilon^3w_3,\ \epsilon^5w_4)$, another imperfect point. By use of $X_1T_3^{(\prime\prime)}X_1^{-1}=T_3^{(\prime\prime\prime)}$, we get $(w_2,\ w_3,\ w_4)\sim(\epsilon^3w_2,\ \epsilon w_3,\ \epsilon^3w_4)$, which indicates that $T_3^{(\prime\prime\prime)}$ is $x_2':x_3':x_4'=\epsilon^2x_2:x_3:\epsilon^2x_4$. Hence we have found a perfect point, and the theorem follows.

THEOREM 6. *Along the invariant direction $x_1=x_4=0$, the invariant imperfect point P_3 has no perfect points in the first- and second-order neighborhoods but does have one in the third-order neighborhood.*

3. *Sections by Sextics.* Consider the complete system of curves $|B|$ cut out on F by all surfaces of order six. Its dimension is 63, its genus is 46, and the number of variable intersections of two members of the system is 108. A curve B of this system is not in general transformed into itself. There are, however, seven partial systems $|B_i|$ in $|B|$ which are transformed into themselves. By use of $|B_1|$ we find

$$\begin{aligned}b_1x_1x_4^5+b_2x_2x_3x_4^4+b_3x_3^3x_4^3+b_4x_1^2x_2^2x_4^2+b_5x_1^3x_3x_4^2+b_6x_2^5x_4\\+b_7x_1x_2^3x_3x_4+b_8x_1^2x_2x_3^2x_4+b_9x_1^2x_3^4+b_{10}x_1x_2^2x_3^3\\+b_{11}x_2^4x_3^2+b_{12}x_1^5x_2=0.\end{aligned}$$

If we refer the curves B_1 projectively to the hyperplanes of a linear space of eleven dimensions, we obtain a surface ϕ. The equations of transformation for mapping I_7 upon ϕ in S_{11} are

$$\begin{array}{lll} \rho X_1 = x_1x_4^5, & \rho X_5 = x_1^3x_3x_4^2, & \rho X_9 = x_1^2x_3^4, \\ \rho X_2 = x_2x_3x_4^4, & \rho X_6 = x_2^5x_4, & \rho X_{10} = x_1x_2^2x_3^3, \\ \rho X_3 = x_3^3x_4^3, & \rho X_7 = x_1x_2^3x_3x_4, & \rho X_{11} = x_2^4x_3^2, \\ \rho X_4 = x_1^2x_2^2x_4^2, & \rho X_8 = x_1^2x_2x_3^2x_4, & \rho X_{12} = x_1^5x_2. \end{array}$$

By eliminating ρ, x_1, x_2, x_3, x_4 from these twelve equations and from $F_3(x_1,x_2,x_3,x_4)=0$, we get as the nine equations defining the surface

$$\left\| \begin{matrix} X_1 & X_4 & X_7 \\ X_2 & X_7 & X_{11} \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} X_4 & X_7 & X_6 \\ X_8 & X_{10} & X_{11} \end{matrix} \right\| = 0,$$

$$\left\| \begin{matrix} X_7 & X_{10} & X_6 \\ X_8 & X_9 & X_7 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} X_5 & X_2 & X_3 \\ X_{12} & X_4 & X_8 \end{matrix} \right\| = 0,$$

and $aX_7+bX_8+cX_4=0$.

All the curves B_1 pass through the invariant points P_1, P_2, P_3, P_4. Consider point P_2. Its tangent plane is $x_3=0$. It cuts the sextic surfaces in the curves

$$x_3 = 0, \; b_1x_1x_4^5 + b_4x_1^2x_2^2x_4^2 + b_6x_2^5x_4 + b_{12}x_1^5x_2 = 0.$$

This curve passes simply through P_2 along the invariant direction $x_3=x_4=0$. If $b_6=0$, the curve degenerates into a line and a quintic with a triple point formed by a simple branch passing through a cusp. The line is the simple tangent at the triple point, while the cuspidal tangent cuts the curve again at an undulation. If $b_6=b_4=0$, the curve degenerates into a quintic with a five-fold point (having five coincident tangents) and a line, the tangent at the five-fold point. If $b_6=b_4=b_{12}=0$, P_2 is a six-fold point. The curve breaks up into six lines, one of which is $x_1=x_3=0$, while the other five are $x_3=x_4=0$ counted five times. Thus the system of B_1 curves passes through P_2 along invariant directions.

The tangent plane at P_3 is $x_1=0$. It cuts the sextic surfaces in the curves $x_1=0$, $b_2x_2x_3x_4^4+b_3x_3^3x_4^3+b_6x_2^5x_4+b_{11}x_2^4x_3^2=0$. P_3 is a special triple point on these curves having three coincident tangents. The curves touch at P_3 along the invariant direction $x_1=x_4=0$. If $b_3=0$, then the curve passes through P_3 four times and is tangent to $x_1=x_2=0$ four times. When $b_3=b_{11}=0$, the curve is five fold at P_3, passing through along the invariant direction $x_1=x_2=0$ once and along $x_1=x_4=0$ four times. When $b_3=b_{11}=b_2=0$, the curve breaks up into six lines through P_3, namely, $x_1=x_4=0$ counted simply and $x_1=x_2=0$ counted five times. The following theorem may be stated.

THEOREM 7. *The* $|B_1|$ *curves pass through the imperfect points only along the invariant directions.*

4. *Sections by Quintics.* The complete system of curves $|C|$ cut out on F by all surfaces of order five has dimension 45, genus 31, and the number of variable intersections of two members of the system is 75. There are seven partial systems $|C_i|$ in $|C|$ which are transformed into themselves. By the use of $|C_1|$ we find

$$c_1x_1^2x_4^3 + c_2x_1x_2x_3x_4^2 + c_3x_2^3x_4^2 + c_4x_1x_3^3x_4 + c_5x_2^2x_3^2x_4 + c_6x_2x_3^4 + c_7x_1^4x_3 + c_8x_1^3x_2^2 = 0.$$

By referring the curves C_1 projectively to the hyperplanes of a linear space of seven dimensions, we obtain a surface ϕ. The equations of transformation for mapping I_7 upon ϕ in S_7 are

$$\rho X_1 = x_1^2x_4^3, \quad \rho X_3 = x_2^3x_4^2, \quad \rho X_5 = x_2^2x_3^2x_4, \quad \rho X_7 = x_1^4x_3,$$
$$\rho X_2 = x_1x_2x_3x_4^2, \quad \rho X_4 = x_1x_3^3x_4, \quad \rho X_6 = x_2x_3^4, \quad \rho X_8 = x_1^3x_2^2.$$

Eliminate ρ, x_1, x_2, x_3, x_4 from these eight equations and from $F_3(x_1, x_2, x_3, x_4)=0$. The five equations defining the surface ϕ are

$$\left\|\begin{matrix} X_1 & X_2 & X_4 \\ X_2 & X_5 & X_6 \end{matrix}\right\| = 0, \quad \left\|\begin{matrix} X_3 & X_5 \\ X_5 & X_6 \end{matrix}\right\| = 0, \quad \left\|\begin{matrix} X_2 & X_7 \\ X_3 & X_8 \end{matrix}\right\| = 0,$$

and

$$aX_5 + bX_4 + cX_2 = 0.$$

All the curves C_1 pass through the invariant points P_1, P_2, P_3, P_4. Its tangent plane at P_2 is $x_3=0$. It cuts the quintic surfaces in the curves $x_3=0$, $c_1x_1^2x_4^3+c_3x_2^3x_4^2+c_8x_1^3x_2^2=0$. This curve has a double point at P_2, both branches being tangent to the invariant direction $x_3=x_4=0$. When $c_3=0$, the curve degenerates into a cuspidal cubic and a repeated line (the flex tangent, which is $x_1=x_3=0$). If $c_3=c_8=0$, the curve degenerates into five straight lines through P_2. They are $x_1=x_3=0$ counted twice and

$x_3=x_4=0$ counted three times. Hence these curves pass through P_2 along invariant directions.

The tangent plane to F_3 at P_3 is $x_1=0$. It cuts the quintic surfaces in the curves $x_1=0$, $c_3x_2^3x_4^2+c_5x_2^2x_3^2x_4+c_6x_2x_3^4=0$. This curve is simple at P_3, passing through along the invariant direction $x_1=x_2=0$. When $c_6=0$, the curve degenerates into a conic and three lines, one line $(x_1=x_4=0)$ a simple tangent and the other $(x_1=x_2=0)$ a repeated tangent. If $c_6=c_5=0$, then the curve breaks up into the line $x_1=x_2=0$ counted three times and the line $x_1=x_4=0$ counted twice.

THEOREM 8. *The* $|C_1|$ *curves pass through imperfect points only along invariant directions.*

5. *Sections by Quartics.* The dimension of the complete system of curves $|D|$ cut out on F by all surfaces of order four is 30, its genus is 19, and the number of variable intersections of two members of the system is 48. By use of $|D_1|$ we find

$$d_1x_2x_4^3 + d_2x_3^2x_4^2 + d_3x_1^3x_4 + d_4x_1x_2^3 + d_5x_1^2x_2x_3 = 0.$$

We refer the curves D_1 projectively to the hyperplanes of a linear space of four dimensions, and obtain a surface ϕ. The equations of transformation for mapping I_7 upon ϕ in S_4 are

$$\rho X_1 = x_2x_4^3,\ \rho X_2 = x_3^2x_4^2,\ \rho X_3 = x_1^3x_4,\ \rho X_4 = x_1x_2^3,$$
$$\rho X_5 = x_1^2x_2x_3.$$

The two equations defining the surface are

$$X_1X_5^2 = X_2X_3X_4 \qquad \text{and} \qquad aX_1X_5 + bX_2X_3 + cX_1X_3 = 0.$$

All the curves D_1 pass through the invariant points P_1, P_2, P_3, and P_4. Consider the point P_2. Its tangent plane is $x_3=0$. It cuts the quartic surfaces in the curves

$$x_3 = 0,\ d_1x_2x_4^3 + d_3x_1^3x_4 + d_4x_1x_2^3 = 0.$$

This curve passes simply through P_2 along the $x_1=x_3=0$ direction. When $d_4=0$, the curve degenerates into a cuspidal cubic and the cusp tangent $(x_3=x_4=0)$. When $d_4=d_1=0$, the curve breaks up into four lines through P_2. They are $x_1=x_3=0$ counted three times and $x_3=x_4=0$ counted once.

The tangent plane at P_3 cuts the quartic surfaces in the curves $x_1=0$, $d_1x_2x_4^3+d_2x_3^2x_4^2=0$. The quartic curves degenerate into conics and a repeated (tangent) line. When $d_2=0$, they break up into $x_1=x_4=0$ counted three times and $x_1=x_2=0$ counted once.

THEOREM 9. *The* $|D_1|$ *curves pass through imperfect points only along invariant directions.*

6. *Sections by Cubics.* Investigate the complete system of curves $|E|$ cut out on F by all surfaces of order three. Its dimension is 19, genus is 10, and the number of variable intersections of the two members of the system is 27. The use of $|E_1|$ gives $e_1x_2^2x_3+e_2x_1x_3^2+e_3x_1x_2x_4=0$. The equations of transformation for referring the curves E_1 projectively to the lines of a plane are $\rho X_1=x_2^2x_3$, $\rho X_2=x_1x_3^2$, $\rho X_3=x_1x_2x_4$. A curve $aX_1+bX_2+cX_3=0$ is obtained, instead of a surface. All the curves E_1 pass through the invariant points P_1, P_2, P_3, and P_4. The tangent plane to F_3 at P_2 is $x_3=0$. This intersects E_1 surfaces in $x_3=0$, $x_1x_2x_4=0$. Hence, the cubic curve becomes three straight lines, two of which pass through P_2, namely, $x_1=x_3=0$ and $x_3=x_4=0$. At P_3 the tangent plane is $x_1=0$. It cuts the E_1 surfaces in $x_1=0$, $x_2{}^2x_3=0$. This degenerate cubic curve also has a double point at P_3. The branches are $x_1=x_2=0$ counted twice. Hence the following theorem is proved.

THEOREM 10. *The system of invariant curves cut out upon F by surfaces of degree lower than seven all pass through the coincidence points along the invariant directions. The number of branches through each point is less than seven.*

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