RADÓ ON PLATEAU'S PROBLEM

On the Problem of Plateau. By Tibor Radó. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Heft 2. Berlin, Springer, 1933. 109 pp.

This report is an interesting and informative account of the recent work on the problem of Plateau—the problem which requires a minimal surface, or surface of least area, bounded by a given closed curve—dealing especially with the contributions made in recent years by Garnier, Douglas, Radó, and McShane. The survey consists of six chapters.

Chapter 1 reviews some general notions concerning length and area, with the implications of the various definitions that have been proposed for the latter. The definition of area most significant for the work on the Plateau problem is that given by Lebesgue in his thesis: the area of a surface S is the *least* limit towards which the area of a polyhedral surface P can tend when P approaches to S. This definition is associated with the essential lower semi-continuity property of area.

The importance for the problem of Plateau of monotonic transformations of the unit circumference C into the given contour Γ is pointed out, that is, the totality of ways of representing Γ as a topological image of C. The introduction of these monotonic transformations into the problem, as well as the important remark of the *compactness* of the totality of them, is due to the present reviewer in his abstracts in this Bulletin and his European lectures of 1929. This property of compactness makes it possible to prove the attainment of the minimum of any lower semi-continuous functional having these monotonic transformations as range of the argument, by a simple and classical procedure going back to Weierstrass and Fréchet. The totality of surfaces S bounded by a given contour does not form a compact set; it was the necessity of obtaining such a set as range of the argument of the functional, area of S, that underlay the restrictions to which the contour was subjected in the earlier work, notably that of Lebesgue and Haar.

In Chapter 2 are reviewed some facts from the differential geometry of minimal surfaces, necessary in the sequel, especially the formulas of Monge, Weierstrass, and Schwarz.

Chapter 3 begins with five modes of formulation $P_1, \cdots, P_5$ of the Plateau problem, varying somewhat in their implications. In these formulations a minimal surface is taken to be one defined by the Weierstrass formulas

$$x_i = \Re \phi_i(w), \quad \sum_{i=1}^{3} \phi_i'^2(w) = 0,$$

which amount to the condition of zero mean curvature for the surface together with the requirement of its conformal mapping on a circular disc.

If we require a surface which not only is minimal in the sense of the preceding formulas, but also has least area, then we have what Radó calls the "simultaneous problem." Examples, due to Schwarz, are cited of minimal surfaces which do not have the least area property in their entire extent.

The importance of singular or branch points for the formulation and solution of the problem is emphasized. A branch point is one where $\phi_i'(w)=0$, for $i=1, 2, 3$, and is said to be of nth order if the derivatives of each ϕ_i up to the nth order inclusive vanish, while the same is not true for the $(n+1)$th order.

An interesting theorem, due to Radó, is that if the contour Γ has a simply-covered *star-shaped* central or parallel projection on some plane, then a minimal surface bounded by Γ cannot have branch points. A plane curve is called star-shaped if there exists a point O in its plane such that every ray issuing from O intersects the curve in one and only one point. All convex curves are star-shaped, but not conversely.

The various modes of formulation of the Plateau problem lead to various uniqueness questions, in answer to which only some partial results have been obtained. Radó has proved the theorem: if Γ has a simply-covered *convex* central projection on some plane, then there is one and only one minimal surface bounded by Γ. A consequence is that the minimal surface bounded by any given skew quadrilateral is unique, a theorem going back to Schwarz.

That a given contour may bound several distinct minimal surfaces, of the topological type of a circular disc, is shown by examples due to Schwarz, Radó, and Wiener. In the series of statements $P_1, \cdots, P_5$ of the Plateau problem, the first four assume the surface in parametric form, while P_5 requires the surface in the form $z=f(x, y)$ and gives the contour with a simply-covered orthogonal projection on the xy-plane. Chapter 4 deals with this non-parametric form of the problem, where, besides, Γ is subject to a certain "three-point condition," whose chief implication is that the orthogonal projection of Γ on the xy-plane is convex. Chapter 5 begins the treatment of the recent work, where the contour has a general shape, and the surface is assumed in parametric form. The solutions of Garnier, Douglas, and Radó are discussed.

Garnier's result is subject to the restriction that the contour *shall have no knots*, and shall consist of a finite number of arcs each with bounded curvature. Douglas' result, the most general, allows Γ to be any Jordan curve whatever, and indeed, in euclidean space of an arbitrary number n of dimensions, the value of n having no influence on either method or result.

Radó's result restricts the contour to be capable of bounding at least one surface of finite area. This condition will not be satisfied if the contour contains a spiral which contracts with a sufficient degree of slowness, for instance, the analytic curve (given by the reviewer†) which is defined in spherical polar coordinates by the equations $r=\cos\phi$, $\theta=\tan^6\phi$, (ϕ latitude, θ longitude).

As to method, Garnier follows the classic program of Riemann and Weierstrass, involving solution of the so-called monodromy group problem in the theory of linear differential equations of the second order.

Douglas' method is based on the functional $A(g)$ introduced by him, whose argument g is an arbitrary representation $x_i=g_i(\theta)$ of Γ as topological image of the unit circumference C, and whose definition is

$$A(g) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{l^*}{l}\right)^2 d\theta d\phi,$$

† J. Douglas, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 448–451.

where l^*, l are the lengths of corresponding chords of Γ and C. The function $A(g)$ is amenable to immediate treatment by the same method which Weierstrass gave to prove the attainment of the minimum of a continuous function of a real variable. As brought out by Fréchet in his thesis, this reasoning requires only the two postulates that the function be lower semi-continuous and its argument range compact, and both these postulates are satisfied by $A(g)$.

Radó's method is based on constructing a sequence of polyhedral surfaces whose boundaries tend to Γ and whose areas tend to $m(\Gamma)$, the minimum of simply-connected areas bounded by Γ, and then representing these polyhedra conformally on the unit circular disc. Thus Radó's solution presumes the theory of conformal mapping.

On the other hand, it is an essential feature of the work of Douglas that when $n=2$ and Γ becomes a plane curve, his method gives a conformal map of the interior of Γ on the unit circular disc, attaching continuously to a topological correspondence between Γ and the unit circumference, that is, an independent proof of the classic Riemann mapping theorem together with the boundary correspondence theorem of Osgood and Carathéodory.

Chapter 6 deals first with the simultaneous problem concerned with the least area property of the minimal surface whose existence is established by the preceding methods. The contributions of Radó and Douglas are described, followed by a discussion of the new work of McShane, one of whose essential features is to operate only in the class of saddle-surfaces and to utilize certain selection theorems applying to such surfaces.

The report concludes with an account of the work of Douglas on the minimal surface bounded by two given non-intersecting contours, and on one-sided minimal surfaces with a given bounding curve.

In the latter theory, one of the theorems proved by the present reviewer is that if the simply-connected minimal surface determined by Γ has in its interior a branch point like that of the Riemann surface for the $2n$th root of z, then a one-sided minimal surface bounded by Γ exists. The published paper† containing this theorem includes a diagram by way of illustration. Radó attributes to the reviewer the inference that every contour whose *orthogonal projection* has the general shape of this illustration will give a simply-connected minimal surface with a branch point like that of $z^{1/2}$. This inference, together with the difficulties to which it leads, is nowhere stated nor implied in the reviewer's text.

Some corrections are as follows. Page 104, third line from bottom, instead of $4\pi\alpha^2$ read $\pi\alpha^2$. The same correction in formula (6.17) on page 105. In formulas (6.18), (6.19), strike out the term $3\pi\alpha^2$. Page 79, last line, write $i\lambda$ in place of λ. Corresponding to the two transformations at the bottom of page 79, $A(T)$ gives two functions $J_1(\lambda)$, $J_2(\lambda)$. In the second displayed formula on page 80, replace $J'(0)$ by $\frac{1}{2}[J_1'(0)-iJ_2'(0)]$.

The tract is clearly written, contains much information presented in a compact style, and should stimulate interest in its important field, where much valuable work still remains to be done.

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† J. Douglas, *One-sided minimal surfaces with a given boundary*, Transactions of this Society, vol. 34 (1932), pp. 731–756; diagram on p. 733.