

SOME DEFINITE INTEGRALS INVOLVING
SELF-RECIPROCAL FUNCTIONS

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1. *Introduction.* In one of his papers, Ramanujan* has proved formally that if

$$\phi_{\omega}(t) = \int_0^{\infty} \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi \omega x^2} dx,$$

then

$$(1) \quad \phi_{\omega}(t) = \frac{e^{-\pi t^2 / (4\omega)}}{\omega^{1/2}} \phi_{1/\omega}\left(\frac{it}{\omega}\right).$$

An examination of the proof shows that it rests on the fact that $\operatorname{sech}[x(\pi/2)^{1/2}]$ is self-reciprocal for cosine-transforms. The present investigation was suggested by this fact. The object of this note is to obtain a generalization of (1).

Following Hardy and Titchmarsh, I will say that a function is R , if it is its own J , transform, and it is $-R$, if it is skew-reciprocal for J , transforms; also, for $R_{1/2}$ and $R_{-1/2}$, I will write R_s and R_c , respectively.

2. **THEOREM 1.** *If*

$$\phi_{\omega}(t) = \omega^{1/2} \int_0^{\infty} e^{-\omega^2 x^2 / 2} f(x) \cos t \omega x dx,$$

where $f(x)$ is R_c and is such that $\int_0^{\infty} |f(x)| dx$ converges, then

$$(2) \quad \phi_{\omega}(t) = e^{-t^2 / 2} \phi_{1/\omega}(it).$$

We have

$$(3) \quad \phi_{\omega}(t) = \left(\frac{2\omega}{\pi}\right)^{1/2} \int_0^{\infty} e^{-\omega^2 x^2 / 2} \cos t \omega x dx \int_0^{\infty} f(y) \cos xy dy.$$

This double integral is absolutely convergent, as we see by comparison with

* Ramanujan, *Some definite integrals*, Collected Papers, Cambridge University Press, 1927, pp. 202–207.

$$\int_0^\infty e^{-\omega^2 x^2/2} dx \int_0^\infty |f(y)| dy.$$

Hence we may invert the order of integration in (3). Thus

$$\begin{aligned} \phi_\omega(t) &= \left(\frac{2\omega}{\pi}\right)^{1/2} \int_0^\infty f(y) dy \int_0^\infty e^{-\omega^2 x^2/2} \cos t\omega x \cos xy dx \\ &= \left(\frac{\omega}{2\pi}\right)^{1/2} \int_0^\infty f(y) dy \int_0^\infty e^{-\omega^2 x^2/2} \{ \cos (y + t\omega)x \\ &\quad + \cos (y - t\omega)x \} dx \\ &= \frac{1}{2\omega^{1/2}} \int_0^\infty f(y) \{ e^{-(y+t\omega)^2/(2\omega^2)} + e^{-(y-t\omega)^2/(2\omega^2)} \} dy \\ &= \frac{1}{2\omega^{1/2}} \int_0^\infty e^{-y^2/(2\omega)^2 - t^2/2} (e^{-yt/\omega} + e^{yt/\omega}) f(y) dy \\ &= \frac{e^{-t^2/2}}{\omega^{1/2}} \int_0^\infty e^{-y^2/(2\omega^2)} \cosh \frac{yt}{\omega} f(y) dy, \end{aligned}$$

which establishes (2). As an illustration, (2) may be verified for $f(x) = e^{-x^2/2}$.

3. THEOREM 2. If

$$\psi_\omega(t) = \omega^{1/2} \int_0^\infty e^{-\omega^2 x^2/2} f(x) \sin t\omega x dx,$$

where $f(x)$ is R_s and is such that $\int_0^\infty |f(x)| dx$ converges, then

$$(4) \quad \psi_\omega(t) = -ie^{-t^2/2} \psi_{1/\omega}(it). \dagger$$

This can be proved in exactly the same way as Theorem 1. To illustrate this theorem, (4) may be verified for $f(x) = xe^{-x^2/2}$.

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† Theorems 1 and 2 themselves depend upon the fact that $e^{-x^2/2}$ is R_c , and can be further generalized, but the generalized theorems do not seem to be very useful.