nacci series 0, 1, 2, 3, 5, 8, 13, · · · giving the values of the Lucas function \( U_n \) associated with the polynomial \( x^2 - x - 1 \). This polynomial is irreducible modulo 13, so that the period of the Fibonacci series modulo 13 gives the period of the mark \( \alpha \) associated with \( x^2 - x - 1 \) in the finite field of order 13. We have \( \omega = 7, \) norm \( \alpha = -1, \theta = 2, k = 2, \sigma = 2, \ p - 1 = 12. \) Hence (2) becomes \( (2, 2) | \delta | (2, 12), \) so that \( \delta = 2. \) Hence the period is 28, which is easily verified directly. It seems quite difficult to determine the exact value of \( \delta \) in all cases.*

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**ON A PROBLEM OF KNASTER AND ZARANKIEWICZ†**

**BY J. H. ROBERTS**

Knaster and Zarankiewicz have proposed the following problem:‡ "Does every continuum \( A \) contain a subcontinuum \( B \) such that \( A - B \) is connected?" Knaster has shown,§ by an example in 3-space, that the answer is in the negative. In the present paper an example is given of a plane continuum \( M \) such that every non-degenerate proper subcontinuum of \( M \) disconnects \( M \).

The point sets considered in this paper all lie in a plane.

**DEFINITION OF** \( F(C; X, Y; \epsilon) \). Let \( C \) be any simple closed curve, \( X \) and \( Y \) distinct points of \( C \), and \( \epsilon \) any positive number. There exists a finite set of points \( A_1, A_2, \ldots, A_n \) \((n > 2)\), such that (a) \( A_1 + A_2 + \cdots + A_n \) contains \( X + Y \), (b) \( A_1, A_2, \ldots, A_n \) lie on \( C \) in the order \( A_1A_2 \cdots A_nA_1 \), and (c) \( A_1 \) and \( A_{i+1} \) (subscripts are to be reduced modulo \( n \)) are the end points of an arc \( t_i \) of diameter <\( \epsilon \) which is a subset of \( C \) not containing \( A_{i+2} \). There exists a set of mutually exclusive arc segments \( v_1, v_2, \ldots, v_n \) lying within \( C \) such that \( v_i + t_i \) is a simple closed curve \( w_i \) of diameter <\( \epsilon \). Let \( J \) denote the simple closed curve

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* See the discussion at the close of my paper, Transactions of this Society, vol. 33 (1931), p. 165.
† Presented to the Society, December 1, 1933.
‡ Fundamenta Mathematicae, vol. 8 (1926), Problem 42, p. 376.
There exist infinite sequences of simple closed curves $C_{ij}$, ($i=1, 2, \cdots, n$; $j=1, 2, \cdots$), such that (1) $C_{ij}$ contains $A_i$ but otherwise lies within $J$, (2) the sequence $C_{i1}, C_{i2}, C_{i3}, \cdots$ has as sequential limit set the arc $A_i + v_i + A_{i+1}$, (3) $C_{ij}$ is of diameter $< \epsilon$, (4) $C_{ik} \cap C_{ih} = A_i$ ($i \neq k$), and $C_{ij}, C_{ih} = 0$, ($i \neq h$), and (5) no point of $C_{ij}$ lies within any $C_{ih}$. The set $F(C; X, Y; \epsilon)$ is defined as the sum of all the curves $C_{ij}$ and the $n$ curves $w_i$:

$$F(C; X, Y; \epsilon) = \sum_{i=1}^{n} \left[ w_i + \sum_{j=1}^{\infty} C_{ij} \right].$$

**Definition of $M$.** Let $E$ be any simple closed curve, $X$ and $Y$ any two points of $E$. Let $K_1$ denote a set $F(E; X, Y; 1)$. Then $K_1 = \sum_{i=1}^{\infty} E_{i,0}$, where for each $i$, $E_{i,0}$ is a simple closed curve of diameter $< 1$, and the common part of $E_{i,0}$ and the sum of the other curves $E_{1,1}, E_{2,1}, \cdots$ either is one point, or is two points. Thus $E_{i,0}$ contains distinct points $X_{i,0}$ and $Y_{i,0}$ such that no other point of $E_{i,0}$ belongs to $E_{i,j}$, ($i \neq j$). For each $i$ let $G_{i,1}$ be a set $F(E_{i,1}; X_{i,1}, Y_{i,1}; 1/2)$ and let $K_2$ be $G_{1,1} + G_{1,2} + \cdots$.

Suppose $K_1, K_2, \cdots, K_n$, ($n > 1$), have been defined, $K_1$ being as defined above and, for each $i$, the following properties obtain:

I. $K_i$ is the sum of a countable set of simple closed curves $E_{1,1}, E_{2,1}, \cdots$.

II. Each curve $E_{i,h}$ has, in common with the sum of the other curves $E_{1,h}, E_{2,h}, \cdots$, either one point or two points.

III. $X_{i,h}$ and $Y_{i,h}$ are distinct points of $E_{i,h}$ such that no other point of $E_{i,h}$ belongs to the sum of the other curves $E_{1,h}, E_{2,h}, \cdots$.

IV. No point is common to the interiors of two curves $E_{i,h}$ and $E_{i,k}$, ($h \neq k$).

V. $K_{i+1}$, ($i < n$), is a subset of the sum of $K_i$ and the interiors of all the curves $E_{1,1}, E_{2,1}, \cdots$.

VI. The subset of $K_{i+1}$, ($i < n$), which lies on and within $E_{i,h}$ is a set $F(E_{i,h}; X_{i,h}, Y_{i,h}; 1/[i+1])$.

For $n = 2$, the sets $K_1$ and $K_2$ defined above have these properties. For each $i$, ($i \leq n$), let $U_i$ be the set of all points of $K_i$ each of which belongs to at least two curves of the set $E_{1,1}, E_{2,1}, \cdots$, and let $D_i$ denote $K_i$ plus the interiors of all the curves $E_{1,1}, E_{2,1}, \cdots$. 

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For each \( k \) let \( G_{nk} \) be a set \( F(E_{nk}; X_{nk}, Y_{nk}; 1/[n+1]) \), and let 
\( K_{n+1} = G_{n1} + G_{n2} + \cdots \). Then it readily follows that the sequence 
\( K_1, K_2, \cdots, K_n, K_{n+1} \) has the properties I–VI above. 
Hence there is an infinite sequence \( K_1, K_2, \cdots \) with properties 
I–VI; \( K_1 \) being a set \( F(E; X, Y; 1) \). Let \( M \) be \( K_1 + K_2 + \cdots \) 
plus all limit points. This is the same as the common part of 
\( D_1, D_2, \cdots \).

**Proof that** \( M - H \) **is not connected.** Suppose \( H \) is a non-
degenerate proper subcontinuum of \( M \). Suppose \( M - H \) is connected. 
Now the components of \( M - U_n \) are of diameter \(< 1/n \).
Hence there exists an \( n \) such that \( H \) contains a point \( P \) of \( U_n \).
It will be shown that if \( H \) contains a point of \( U_n \), then it contains all of \( U_n \). In view of this, and the fact that \( U_n \) is a subset 
of \( U_{n+1} \) and that \( M = (U_1 + U_2 + \cdots) \) plus limit points, it follows that \( H = M \), which is a contradiction.

It remains to show that if \( H \) contains a point \( P \) of \( U_n \), then it contains all of \( U_n \). Let \( h \) be such that \( P \) belongs to \( E_{nh} \). The subset of \( K_{n+1} \) which lies on and within \( E_{nh} \) is a set \( F(E_{nh}; X_{nh}, Y_{nh}; 1/[n+1]) \). The points of \( U_{n+1} \) in this set can be labeled \( B_1, B_2, \cdots, B_k \), so that they lie on \( E_{nh} \) in the order 
\( B_1B_2\cdots B_kB_1 \). Now each of the infinity of components of 
\( K_{n+1} - B_i \) is a subset of a different component of \( M - B_i \). Hence 
if \( H \) contains \( B_i \), and \( M - H \) is connected, \( H \) must contain all 
save one of these components. But \( B_{i+1} \) is a limit point of the sum of the components of \( K_{n+1} - B_i \). Hence, if \( H \) contains \( B_i \), it contains \( B_{i+1} \). But for some \( i \), \( P = B_i \). Thus \( H \) contains all the points of \( U_{n+1} \) on \( E_{nh} \), and therefore the one or two points of \( U_n \) on \( E_{nh} \). Now any two curves \( E_{nh} \) and \( E_{nk} \), of the set 
\( E_{n1}, E_{n2}, \cdots \), can be joined by a finite chain \( L_1, L_2, \cdots, L_e \) of 
curves of the set \( E_{n1}, E_{n2}, \cdots, L_1 \) having a point in common 
with \( E_{nh}, L_i \) having a point in common with \( L_{i+1}, (i < e) \), and \( L_e \) 
having a point in common with \( E_{nk} \). Since these common points are in \( U_n \), and \( H \) contains a point of \( U_n \) in \( E_{nh} \), it readily follows, 
by repeated application of the above argument, that \( H \) contains every point of \( U_n \) in \( E_{nh} + L_1 + L_2 + \cdots + L_e + E_{nk} \), and therefore \( H \) contains every point of \( U_n \).

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