

THE PROBABILITY LAW FOR THE SUM OF n
INDEPENDENT VARIABLES, EACH SUBJECT
TO THE LAW $(1/(2h))\operatorname{sech}(\pi x/(2h))^*$

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1. *Introduction.* Let the probability of selecting the chance real variable x from the interval $(x, x+dx)$ be to within infinitesimals of higher order, the quantity $(1/(2h)) \operatorname{sech}(\pi x/(2h))dx$. This hyperbolic secant probability or frequency function has been used by others. Roa considered this function in many details as a generating function for frequency functions and gave numerical tables pertaining to it.† Fisher obtained as a special case a type of this frequency law for the frequency of the “intra-class” correlation coefficient.‡ Dodd investigated this probability function as a particular case when considering measurements under general laws of errors.§ The author obtained the law for the sum of n independent variables when each is subject to this hyperbolic law but was not able to express the sum function without the use of an integral.||

The object of this article is to find the probability function for the sum $\sum_{i=1}^n x_i$ when each variable x_i is subject to the probability function $(1/(2h)) \operatorname{sech}(\pi x_i/(2h))$, or to find the probability to within infinitesimals of higher order that

$$u \leq \sum_{i=1}^n x_i \leq u + du.$$

2. *Case I: n Finite.* If a general method due to Dodd¶ be applied to this hyperbolic secant law, the probability law for the sum of n independent variables is

* Presented to the Society, June 22, 1933.

† E. Roa, *A number of new generating functions with applications to statistics*, Thesis, University of Michigan, 1924.

‡ R. A. Fisher, *On the probable error of a coefficient of correlation deduced from a small sample*, *Metron*, vol. 1 (1920–21), pp. 3–32.

§ E. L. Dodd, *Functions of measurements under general laws of errors*, *Skandinavisk Aktuarietidskrift*, 1922, No. 3, pp. 134–158.

|| W. D. Baten, *Frequency laws for the sum of n variables which are subject to given frequency laws*, *Metron*, vol. 10 (1932), No. 3, pp. 75–91.

¶ E. L. Dodd, *The frequency law of a function of variables with given frequency laws*, *Annals of Mathematics*, (2), vol. 27 (1925–26), p. 13.

$$p_n(u) = 2^n h^{-1} \pi^{-1} \int_0^{\infty} (e^x + e^{-x})^{-n} \cos(ux/h) dx.$$

The remainder of this section will be devoted to evaluating this definite integral for even and odd values of n . In order to make this evaluation clearer for any value of n let us first consider the case when n is equal to 4. The sum function is

$$p_4(u) = 2^4 h^{-1} \pi^{-1} \int_0^{\infty} (e^x + e^{-x})^{-4} \cos(ux/h) dx,$$

which can be found by integrating $F_4(z) = e^{iuz/h}(e^z + e^{-z})^{-4}$ around the contour C consisting of the following lines:

- (a) the x -axis from $-R$ to $+R$, where R is large,
- (b) the lines $z = \pm R + yi$,
- (c) the line $z = \pi i + x$.

The only pole within the contour C is $(\pi i/2)$. By Cauchy's residue theorem, we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_C (e^z + e^{-z})^{-4} e^{iuz/h} dz \\ &= \frac{1 - e^{-\pi u/h}}{2\pi i} \int_{-R}^R (e^x + e^{-x})^{-4} e^{iux/h} dx \\ (1) \quad &+ \frac{1}{2\pi i} \int_{\pi i}^0 (e^{-R+yi} + e^{R-yi})^{-4} e^{iu(-R+yi)/h} i dy \\ &+ \frac{1}{2\pi i} \int_0^{\pi i} (e^{R+yi} + e^{-R-yi})^{-4} e^{iu(R+yi)/h} i dy \\ &= \text{the residue at } (\pi i/2), \end{aligned}$$

since the integral of $F_4(z)$ exists for h and u real quantities. The last two integrals approach zero as R becomes infinite, for

$$\lim_{R \rightarrow \infty} F_4(\pm R + yi) = 0.$$

The residue at $z = \pi i/2$ is also the coefficient of z^{-1} in the Laurent expansion of $F_4(z)$ around this point. Let $z = \pi i/2 + w$; then the residue of $F_4(z)$ at $\pi i/2$ is the residue of $F_4(\pi i/2 + w)$

at $w = 0$. The function $F_4(\pi i/2 + w)$ in the neighborhood of $w = 0$ may be written in the form

$$\frac{e^{-\pi u/(2h)}}{(2w)^4} e^{iuw/h} \cdot g_4(w),$$

where

$$g_4(w) = \left(\frac{2w}{e^w - e^{-w}} \right)^4,$$

and, by definition, $g_4(0) = 1/2$. The function $g_4(w)$ is analytic and can be expanded in a Maclaurin series in the neighborhood of the origin, hence

$$g_4(w) = g_4(0) + g_4'(0)w + g_4''(0)w^2/2 + g_4'''(0)w^3/3! + q_4(w),$$

where $q_4(w)$ is the remainder after the fourth term in the Maclaurin series representing $g_4(w)$. To find the coefficient of w^{-1} it is necessary to find the values of the first, second, and third derivatives of $g_4(w)$ at the point $w = 0$. Newsom* obtained the following formula which will be used to find these derivatives at $w = 0$:

$$(2) \quad \left[\frac{d^r}{dw^r} \left(\frac{w}{\sin w} \right)^k \right]_{w=0} = \left[\frac{d^r}{dw^r} \left(\frac{2iw}{e^{iw} - e^{-iw}} \right)^k \right]_{w=0} = \frac{\sum \alpha_1 \alpha_2 \cdots \alpha_r}{{}_{k-1}C_r},$$

in which k is any given positive integer ≥ 2 , $1 \leq r \leq k - 1$, and where $\sum \alpha_1 \alpha_2 \cdots \alpha_r$ denotes the sum of the $\binom{k-1}{r}$ products of r factors each formed by taking the possible combinations of the $(k - 1)$ quantities $\pm(k - 2)i, \pm(k - 4)i, \dots, \left\{ \begin{smallmatrix} \pm i \\ 0 \end{smallmatrix} \right\}$, r at a time; i having the usual interpretation, $i = (-1)^{1/2}$, and where $\left\{ \begin{smallmatrix} \pm i \\ 0 \end{smallmatrix} \right\}$ is understood as $\pm i$ or 0 according as k is odd or even.

Substituting $w = y/i$ in (2), we may write

$$\left[\frac{d^r}{dw^r} \left(\frac{w}{\sin w} \right)^k \right]_{w=0} = \left[\frac{d^r}{dy^r} \left(\frac{2y}{e^y - e^{-y}} \right)^k \right]_{y=0} \cdot \left(\frac{1}{i} \right)^r,$$

* C. V. Newsom, *On the derivatives of $(w/\sin w)^k$ at $w = 0$* , American Mathematical Monthly, vol. 38 (1931), pp. 500-504.

from which the first three derivatives of $g_4(w)$ at $w=0$ can be found. Using these values, we find

$$g_4(w) = [1 - 4w^2/6 + q_4(w)];$$

hence

$$F_4(\pi i/2 + w) = \frac{e^{-\pi u/(2h)}}{2^2 w^2} \left(1 + \frac{i u w}{h} - \frac{u^2 w^2}{2 h^2} - \frac{i u^3 w^3}{3! h^3} + \dots \right) \left[1 - \frac{4 w^2}{6} + q_4(w) \right];$$

and hence the coefficient of w^{-1} is found to be

$$-\frac{i e^{-\pi u/(2h)}}{2^4 \cdot 3! h^3} u(u^2 + 4h^2).$$

By using this residue and by allowing R to become infinite in (1), we find that the probability law for the sum of four variables is

$$p_4(u) = \frac{u \cdot \operatorname{csch}(\pi u/(2h))}{2 \cdot 3! h^4} (u^2 + 4h^2).$$

The probability function

$$p_{2n}(u) = 2^{2n} h^{-1} \pi^{-1} \int_0^\infty (e^x + e^{-x})^{-2n} \cos(ux/h) dx$$

may be obtained in a similar way. To obtain this, it is necessary to find the coefficient of w^{-1} in the Laurent expansion of

$$F_{2n}(\pi i/2 + w) = \frac{e^{-\pi u/(2h)} e^{i u w/h}}{(2w)^{2n}} \left[\sum_{r=0}^{2n-1} g_{2n}^{(r)}(0) \frac{w^r}{r!} + q_{2n}(w) \right],$$

where

$$g_{2n}(w) = \left(\frac{2w}{e^w - e^{-w}} \right)^{2n},$$

and $g_{2n}^{(r)}(0)$ is the r th derivative of $g_{2n}(w)$ at 0, and $q_{2n}(w)$ is the remainder after the $2n$ th term in the Maclaurin series representing $g_{2n}(w)$ in the neighborhood of $w=0$. According to Newton's Theorem, we have

$$F_{2n}(\pi i/2 + w) = \frac{e^{-\pi u/(2h)}}{2^{2n}w^{2n}i^{2n}} \left[1 + \frac{iuw}{h} + \frac{(iuw)^2}{2!h^2} + \dots + \frac{(iuw)^{2n-1}}{(2n-1)!h^{2n-1}} + \dots \right]$$

$$\left[1 + \frac{\sum \alpha_1 \alpha_2}{2_{n-1} C_2} \cdot \frac{w^2}{2} + \frac{\sum \alpha_1 \alpha_2 \alpha_3 \alpha_4}{2_{n-1} C_4} \cdot \frac{w^4}{4!} + \dots + \frac{\sum \alpha_1 \alpha_2 \dots \alpha_{2n-2}}{2_{n-1} C_{2n-2}} \cdot \frac{w^{2n-2}}{(2n-2)!} + q_{2n}(w) \right],$$

from which the coefficient of w^{-1} is found to be

$$\frac{-iue^{-\pi u/(2h)} i^{2n}}{h^{2n-1} 2^{2n} (2n-1)!} \left[u^{2n-2} + h^2 \sum \alpha_1 \alpha_2 u^{2n-4} + h^4 \sum \alpha_1 \alpha_2 \alpha_3 \alpha_4 u^{2n-6} + \dots + h^{2n-2} \sum \alpha_1 \alpha_2 \alpha_3 \dots \alpha_{2n-2} \right].$$

The quantity in brackets is a polynomial in u^2 whose roots are equal to $-(2rh)^2$, where $r = 1, 2, \dots, n-1$. From this residue the probability function for the sum $u = \sum_{i=1}^{2n} x_i$ is found to be

$$p_{2n}(u) = \frac{u \cdot \operatorname{csch}(u\pi/(2h))}{2(2n-1)!h^{2n}} \prod_{r=1}^{n-1} \left[u^2 + (2rh)^2 \right].$$

In a similar manner it can be shown that

$$p_{2n+1}(u) = \frac{\operatorname{sech}(\pi u/(2h))}{2 \cdot h^{2n+1} (2n)!} \prod_{r=0}^{n-1} \left[u^2 + (2r+1)^2 h^2 \right].$$

3. *Case II: n Infinite.* By Liapounoff's theorem* the probability that

$$t_1(2B_n)^{1/2} < u < t_2(2B_n)^{1/2}$$

approaches

$$\pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

* Liapounoff, *Sur une proposition de la théorie des probabilités*, Bulletin de L'Académie de St. Petersburg, (5), vol. 13 (1900), pp. 358-386.

uniformly, where B_n is n times the second moment about the mean of the frequency distribution of the individual variable x , and t_1 and t_2 are any real numbers. The probability that

$$t_1(2B_n)^{1/2} < u < t_2(2B_n)^{1/2}$$

is

$$\int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} p_n(u) du,$$

and hence this expression approaches uniformly

$$\pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

as n approaches infinity, or

$$\lim_{n \rightarrow \infty} \int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} p_n(u) du = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt,$$

or

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} (2B_n)^{1/2} p_n[(2B_n)^{1/2}u] du \\ = \lim_{n \rightarrow \infty} 2hn^{1/2} \int_{t_1}^{t_2} p_n(2hn^{1/2}u) du = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt, \end{aligned}$$

since $(2B_n)^{1/2} = 2hn^{1/2}$. Since the hyperbolic secant law, $(1/(2h)) \operatorname{sech}(\pi x/(2h))$, is of bounded variation and the third moment of the absolute values of the chance variable x is finite, this function, or law, satisfies conditions mentioned by Cramér;* hence, according to Cramér's theorem,

$$2hn^{1/2} p_n(2hn^{1/2}u) \rightarrow \pi^{-1/2} e^{-u^2}.$$

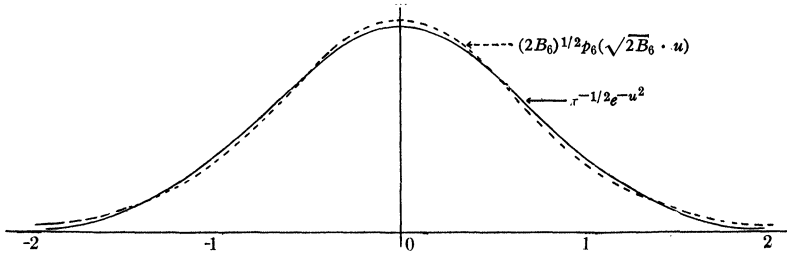
On page 290 are plotted $2h \operatorname{sech}(\pi x/(2h))$ and $\pi^{-1/2} e^{-u^2}$.

* Cramér, H., *On the composition of elementary errors, first paper; Mathematical deductions*, Skandinavisk Aktuarietidskrift, 1928, Nos. 1-2, p. 63.

4. *By-Products.* The function

$$(3) \quad \begin{aligned} & (2B_{2n})^{1/2} p_{2n} [(2B_{2n})^{1/2} \cdot u] \\ &= \frac{2nu \operatorname{csch}(n^{1/2}\pi u)}{(2n-1)!} \prod_{r=1}^{n-1} [4nu^2 + (2r)^2]. \end{aligned}$$

Let n become very large and then substitute zero for u in (3). This should give a value near the value of $\pi^{-1/2} \cdot e^{-u^2}$ at $u=0$; hence



$$\frac{2^{2n}(n!)^2}{(2n)! n^{1/2} \pi} \rightarrow \frac{1}{\pi^{1/2}}, \quad \text{or} \quad \frac{2^{2n}(n!)^2}{(2n)! (\pi n)^{1/2}} \rightarrow 1.$$

Dividing both numerator and denominator by $(2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n)$ and squaring, we find

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \dots \cdot (2n-2)(2n-2)2n}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot \dots \cdot (2n-1)(2n-1)} \rightarrow \frac{\pi}{2},$$

which is a form similar to Wallace’s formula.

When n is odd, a similar expression can be found which leads to this formula of Wallace.

If in $p_{2n}(u)$ and $p_{2n+1}(u)$, u be allowed to be zero, the following definite integrals are evaluated:

$$\int_0^\infty (e^{ht} + e^{-ht})^{-2n} \cos(ut) dt = \frac{1}{2^{2n+1}(2n-1)! h^{2n-1}} \prod_{r=1}^{n-1} (2rh)^2 = \frac{[(n-1)!]^2}{4(2n-1)! h},$$

$$\int_0^\infty (e^{ht} + e^{-ht})^{-2n-1} \cos(ut) dt = \frac{\Gamma[(2n+1)/2]}{\Gamma(n+1) \pi^{1/2} h} \cdot \frac{1}{2^{2n+1}}.$$

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