ENUMERATIVE PROPERTIES OF r-SPACE CURVES*

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In the determination of enumerative properties of algebraic curves it is often convenient to decompose a given curve C^n of order n and genus p to be studied into a number of component curves the sum of whose orders is equal to n. We may decompose C^n in various ways but we find it most convenient to decompose it completely into n lines with $n-1+p$ incidences. We call the system formed by these n lines an n-line or a skew n-sided polygon Γ with $n-1+p$ vertices. To determine the enumerative properties of the given curve C^n, we, in this paper, determine certain enumerative properties of Γ and then interpret the results for C^n. We shall obtain in this manner a number of results for C^n some of which are already well known and the others are less well known or are new.

Let the symbol $\{n\}_{x_1 x_2 \cdots x_q}^{(s)}$ denote the number of groups each consisting of $x_1+x_2+\cdots+x_q$ sides which are arranged in q sets such that each set contains x_i consecutive sides and that any two sets are separated by at least s consecutive sides not contained in them. Thus, $\{n\}_{11}^{(1)}$ means the number of pairs of non-consecutive sides of Γ. If $q=1$, we have $\{n\}_{x_1}^{(s)}$ or just $\{n\}_{x_1}$ which is the number of groups each of x_1 consecutive sides. The symbol $\{n\}^{(s)}$ or $\{n\}$ means the number of groups each containing no members and is therefore equal to unity. Hence,

$$(1) \qquad \{n\}^{(s)} = \{n\} = 1.$$

The following formula can be easily verified or can be proved by the method used below:

$$(2) \qquad \{n\}_{x_1}^{(s)} = \{n\}_{x_1} = n - (x_1 - 1) + (x_1 - 1)p.$$

The number of groups each consisting of q pairs of intersecting sides (or the number of groups of q non-consecutive vertices) of Γ is known† and is given by

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† This result is given without proof by B. C. Wong, *On loci of* $(r-2)$-*spaces incident with curves in r-space*, this Bulletin, vol. 36 (1930), pp. 755–761.

$$(3) \qquad \{n\}^{(0)}_{2_1 2_2 \cdots 2_q} = \sum_{i=0}^{q} \binom{n-q-i}{q-i}\binom{p}{i}.$$

The formula for the number of groups of q non-consecutive sides of Γ is, as we shall see,

$$(4) \qquad \{n\}^{(1)}_{1_1 1_2 \cdots 1_q} = \sum_{i=0}^{t} (-1)^i \binom{n+1-q-2i}{q-2i}\binom{p}{i},$$

where $t=q/2$ for q even and $t=(q-1)/2$ for q odd.

Now we proceed to the determination of $\{n\}^{(s)}_{x_1 x_2 \ldots x_q}$. We may assume

$$\{n\}^{(s)}_{x_1 x_2 \cdots x_q} = M + N - N',$$

where M is the required number when $p=0$ and N and N' are functions of n and p, each containing p as a factor.

To find the value of M, let Γ be an open polygon, that is, a system of n lines $l_1, l_2, \cdots, l_n$ such that l_1 meets l_2, l_2 meets $l_3, \cdots, l_{n-1}$ meets l_n but l_n is skew to l_1. This polygon has $n-1$ vertices, the minimum number it can have without becoming composite. By the simple process of counting, we find

$$M = q!\binom{n - \Sigma x_i - qs + q + s}{q}.$$

This is verified for $n=\Sigma x_i + qs - s$, which gives $M=q!$.

To determine N, we notice that p is the number of additional vertices Γ may possess over and above the minimum, $n-1$. Let a general Γ with $n-1+p$ vertices be given. Consider two series of consecutive sides $a_1, a_2, \cdots$, and $b_1, b_2, \cdots$, respectively. If all the a's are skew to all the b's, then $a_1, a_2, \cdots, a_{x_i}$ and $b_1, b_2, \cdots, b_{x_j}$ form two distinct sets and so do $b_1, b_2, \cdots, b_{x_i}$ and $a_1, a_2, \cdots, a_{x_j}$. Now let a_1 and b_1 meet. The point of meeting is then one of the p additional vertices of Γ over and above the minimum. For each additional vertex we have x_i-1 additional sets given by $a_\lambda, a_{\lambda-1}, \cdots, a_1, b_1, b_2, \cdots, b_{x_i-\lambda}$, $(\lambda=1, 2, \cdots, x_i-1)$. Since each of these sets is to be combined with $q-1$ other sets chosen from the $(n-x_i-2s)$-sided polygon

Γ' obtained from Γ with the x_i+2s consecutive sides $a_{\lambda+s}$, $a_{\lambda+s-1}, \cdots, a_1, b_1, b_2, \cdots, b_{x_i-\lambda+s}$ removed, we have

$$N = \binom{p}{1} \sum_{i=0}^{q} (x_i - 1)\{n - x_i - 2s\}^{(s)}_{x_1x_2\cdots x_q/x_i}.$$

Now it is necessary to deduct a certain number N' of groups arising from the supposition that a_1 and b_1 are incident. Every group containing a pair of sets $a_{\alpha+1}, a_{\alpha+2}, \cdots, a_{\alpha+x_i}$; $b_{\beta+1}$, $b_{\beta+2}, \cdots, b_{\beta+x_j}$, where $0 \leqq \alpha+\beta \leqq s-1$, is to be deducted, as these two sets are separated by only $\alpha+\beta$ sides, namely, a_α, $a_{\alpha-1}, \cdots, a_1, b_1, b_2, \cdots, b_\beta$. These two sets are to be combined with $q-2$ other sets chosen from the $(n-x_i-x_j-2s-\alpha-\beta)$-sided polygon obtained from Γ with the $\alpha+\beta+x_i+x_j+2s$ sides $a_1, a_2, \cdots, a_\alpha, \cdots, a_{\alpha+x_i}, a_{\alpha+x_i+1}, \cdots, a_{\alpha+x_i+s}, b_1$, $b_2, \cdots, b_\beta, \cdots, b_{\beta+x_j}, \cdots, b_{\beta+x_i+s}$ removed. Hence, the number of groups containing these two particular sets is, for all different values of i and j from 1 to q,

$$\sum_{i\neq j}^{q}\{n - x_i - x_j - 2s - \alpha - \beta\}^{(s)}_{x_1x_2\cdots x_q/x_ix_j}.$$

By interchanging the a's and the b's we have twice this number; and by allowing α and β to take on all the permissible values, we have, remembering the factor p,

$$N' = 2\binom{p}{1}\sum_{k=1}^{s} k \sum_{i\neq j}^{q}\{n - x_i - x_j - 2s + 1 - k\}^{(s)}_{x_1x_2\cdots x_q/x_ix_j}.$$

Then, the result for Γ which we started out to derive is the recursion formula

$$\text{(A)}\quad \{n\}^{(s)}_{x_1x_2\cdots x_q} = q!\binom{n - \Sigma x_i - qs + q + s}{q}$$
$$+ \binom{p}{1}\sum_{i=1}^{q}(x_i - 1)\{n - x_i - 2s\}^{(s)}_{x_1x_2\cdots x_q/x_i}$$
$$- 2\binom{p}{1}\sum_{k=1}^{s} k \sum_{i\neq j}^{q}\{n - x_i - x_j$$
$$- 2s + 1 - k\}^{(s)}_{x_1x_2\cdots x_q/x_ix_j}.$$

In the derivation of this formula we have tacitly assumed that all the x's are different. If t of the x's are alike, the right-hand member is to be divided by $t!$. The following result, obtained in somewhat the same manner as above, is more general:

$$\text{(B)}\quad \{n\}^{(s)}_{x_1^{t_1}x_2^{t_2}\cdots x_\nu^{t_\nu}}$$
$$= \frac{(t_1+t_2+\cdots+t_\nu)!}{t_1!t_2!\cdots t_\nu!}\binom{n-\sum t_ix_i - s\sum t_i + \sum t_i + s}{\sum t_i}$$
$$+\binom{p}{1}\sum_{i=1}^{\nu}(x_i-1)\{n-x_i-2s\}^{(s)}_{x_1^{t_1}x_2^{t_2}\cdots x_\nu^{t_\nu}/x_i}$$
$$-\binom{p}{1}\sum_{k=1}^{s}k\sum_{i=1}^{\nu}\{n-2x_i-2s+k-1\}^{(s)}_{x_1^{t_1}x_2^{t_2}\cdots x_\nu^{t_\nu}/x_i^2}$$
$$-2\binom{p}{1}\sum_{k=1}^{s}k\sum_{i\neq j}\{n-x_i-x_j-2s$$
$$+1-k\}^{(s)}_{x_1^{t_1}x_2^{t_2}\cdots x_\nu^{t_\nu}/x_ix_j}.$$

This gives the number of all the groups each consisting of $x_1^{t_1}+x_2^{t_2}+\cdots+x_\nu^{t_\nu}$ sides arranged in $t_1+t_2+\cdots+t_\nu$ sets such that t_1 of these sets contain each x_1 consecutive sides, t_2 contain each x_2 consecutive sides, etc., and such that all the sets are separated from one another by at least s sides.

In the expansion of these formulas, replace $\binom{p}{\lambda}\binom{p}{\mu}$ wherever it appears by $\binom{p}{\lambda+\mu}$. The necessity of this replacement is due to the nature of the combinatory ideas involved or can be easily shown by calculation of known cases.

Putting $s=0$, $\nu=1$, $x_1=2$ in (B) or putting $s=0$, $x_1=x_2=\cdots=x_q=2$ in (A) and dividing by $q!$, we have formula (3). For $s=1$, $x_1=x_2=\cdots=x_q=1$, (A) gives $q!$ times (4), and for $s=1$, $\nu=1$, $x_1=1$, (B) gives exactly the result (4).

Now we are going to derive a few results for curves. Consider a general curve C^n in S_3. Since two skew lines have one apparent intersection or apparent double point, the number of apparent double points of C^n is equal to the number of pairs of non-intersecting sides of Γ and is therefore given by

$$\{n\}^{(1)}_{11}=\binom{n-1}{2}-p.$$

The plane of a conic taken twice may be regarded as its developable. Hence, the order of the developable surface of C^n is twice the number of pairs of consecutive sides of Γ and is, therefore,

$$2\{n\}_2 = 2(n-1+p).$$

Similarly, the developable V_m of C^n in S_r is of order

$$m\{n\}_m = m[n-(m-1)+(m-1)p].$$

Also, the number of hyperosculating hyperplanes of C^n passing through a given point of S_r is

$$r\{n\}_r = r[n-(r-1)+(r-1)p],$$

and the number of hyperstationary tangent S_{r-1}'s is

$$(r+1)\{n\}_{r+1} = (r+1)(n-r+rp).$$

The number of quadrisecant lines of a C^n in S_3 is obtained from the following consideration. Any four general skew lines have two transversals. If they intersect in pairs or if two of them intersect only, there is only one transversal. Hence, the required number is

$$2\{n\}^{(1)}_{1111} + \{n\}^{(1)}_{112} + \{n\}^{(1)}_{22}.$$

It is not difficult to see that the same number is also given by

$$2\binom{n}{4} - \binom{n-2}{2}\{n\}_2 + \{n\}^{(1)}_{22}.$$

Either case yields the same required expression

$$\frac{1}{12}(n-2)(n-3)^2(n-4) - \frac{1}{2}(n-3)(n-4)p + \frac{1}{2}p(p-1).$$

In the same manner, the order of the surface of trisecant lines is found to be

$$2\{n\}^{(1)}_{111} + \{n\}^{(1)}_{12} = \frac{1}{3}(n-1)(n-2)(n-3) - (n-2)p.$$

Since three conics in S_3 not all lying in the same plane have 8 common tangent planes, we say that the number of tritangent planes of C^n in S_3 is 8 times the number of planes each containing three non-consecutive vertices of Γ and is therefore

$$8\{n\}_{222}^{(0)} = 8\left[\binom{n-3}{3} + \binom{n-4}{2}\binom{p}{1} + \binom{n-5}{1}\binom{p}{2} + \binom{p}{3}\right].$$

Similarly, the number of hyperplanes tangent to a C^n in S_r at r distinct points is given by

$$2^r\{n\}_{2^r}^{(0)} = 2^r \sum_{i=0}^{r} \binom{n-r-i}{r-i}\binom{p}{i}.$$

As we have already given examples sufficiently numerous to illustrate the method used, we shall record just one or two more results without explanation, the symbols themselves being indicative of the meaning.

The number of $(r+1)$-secant $(r-1)$-spaces of a C^n in a $2r$-space S_{2r} is

$$\{n\}_{1^{r+1}}^{(1)} = \sum_{i=0}^{t} (-1)^i \binom{n-r-2i}{r+1-2i}\binom{p}{i},$$

where $t=(r+1)/2$ if r is odd and $t=r/2$ if r is even.

As another result we have the order of the V_3 formed by the ∞^1 quinti-secant planes of a C^n in S_4 given by

$$5\binom{n}{5} - 2\binom{n-2}{3}\{n\}_2^{(0)} + \binom{n-4}{1}\{n\}_{22}^{(0)}$$

or

$$5\{n\}_{11111}^{(1)} + 3\{n\}_{2111}^{(1)} + 2\{n\}_{221}^{(1)} + \{n\}_{23}^{(1)} + \{n\}_{311}^{(1)}.$$

The calculation of either yields

$$(n-4)\binom{n-2}{4} - 2\binom{n-3}{3}p + (n-4)\binom{p}{2}.$$

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