THE WORK OF POINCARÉ ON DIFFERENTIAL EQUATIONS

Oeuvres de Henri Poincaré publiées sous les auspices de l'Académie des Sciences par Paul Appell. Tome I publié avec la collaboration de Jules Drach. Paris, Gauthier-Villars, 1928. cxxix+382 pp.

This is the second volume of the Collected Works of Henri Poincaré to appear, the first being volume II, which was published in 1916.* The earlier volume contained his contributions in the general field of automorphic functions, while the new one contains those concerned primarily with ordinary and partial differential equations, and linear difference equations. The order of the material of the Collected Works is to be that contained in Poincaré's own *Analyse* of his papers, and it is decidedly helpful to the reader to find the relevant part of the *Analyse* concerning the material of these first two volumes at the beginning of volume I. A similar plan is to be followed with subsequent volumes.

The starting point of Poincaré's extraordinary mathematical activities is to be found in his *Note sur les propriétés des fonctions définies par les équations différentielles*, published in 1878 in the Journal de l'École Polytechnique. The usual existence theorems for an ordinary differential system of the first order such as

$$(1) \qquad \frac{dx_1}{X_1(x_1, \cdots, x_n)} = \cdots = \frac{dx_n}{X_n(x_1, \cdots, x_n)} = \frac{dt}{1}$$

failed to apply in the neighborhood of the singular points $(x_1, \cdots, x_n)$ at which the functions X_i vanish simultaneously. Briot and Bouquet had examined such singular points in special cases by direct processes; from their work it appeared that integrals of the form $\psi e^{-\lambda t} = \text{const.}$ played an important role; here ψ is an ordinary convergent power series in $x_1, \cdots, x_n$ without constant terms, provided that the singular point in question is taken at the origin of coordinates. But such a function satisfies the associated linear partial differential equation

$$(2) \qquad X_1 \frac{\partial \psi}{\partial x_1} + \cdots + X_n \frac{\partial \psi}{\partial x_n} - \lambda \psi = 0.$$

Poincaré undertook to make this linear partial differential equation the basis of a theory of the solutions of the system (1) near such a singular point.

The outcome is elaborated in his *Thèse* of 1879, *Sur les propriétés des fonctions définies par les équations aux différences partielles*, which almost immediately became classic. A characteristic result is the following: If by a suitable linear transformation of the variables $x_1, \cdots, x_n$ we can reduce the functions X_i to the form $\lambda_i x_i +$ terms of higher degree in $x_1, \cdots, x_n$, with $\lambda_i \neq \lambda_j$ for $i \neq j$ (general case), and if there is a line of the complex λ-plane such that the points λ_i fall on one side of it, while the origin $\lambda = 0$ falls on the other, then the formal series are convergent, and the general integral of (1) near the singular point is furnished by the n equations $\psi_i = ce^{\lambda_i t}$, $(i = 1, \cdots, n)$.

* I reviewed the earlier volume also; see *The work of Poincaré on automorphic functions*, this Bulletin, vol. 26, pp. 161–172.

In case not all of these restrictions are fulfilled, only partial results were obtained. Indeed in the particular case when all the λ_i's are equal the methods of Poincaré fail completely. However, Poincaré characteristically succeeded in exploring various illuminating typical cases. A systematic treatment is lacking even today, although much further progress has been made.

With the local study of the solutions of (1) thus so successfully initiated, Poincaré naturally turned to apply his results to the study of the real solutions in the large, in particular in the simplest case $n=2$ when (1) may be written

$$\frac{dx}{X} = \frac{dy}{Y}, \qquad (x, y, X, Y, \text{ real}). \tag{3}$$

The first three parts of his great memoir *Sur les courbes définies par les équations différentielles*, published in four parts in Liouville's Journal (1881–1886), deal with such an equation in a masterly manner. The fourth part of this memoir deals with the analogous problem for $n=3$. We may refer to an analysis of this work given by Hadamard in his Rice Institute Lectures in 1925 just published,* where the reader will also find reference to the subsequent developments in this field. Fundamental in Poincaré's treatment of the case $n=2$ is the geometric representation of (x, y) as a point of the plane or on a surface, while the equation (3) determines the direction of the corresponding integral curve. This direction becomes indeterminate at the singular points $X_i=0$, $(i=1, \cdots, n)$, referred to above.

The first three parts of this memoir of Poincaré may be regarded as fundamental for the theory of the simplest type of real systems of differential equations, namely for a single equation of the first order. Indeed they are likely to stand always as the most important works in this field. Lagrange, Jacobi, and other mathematicians had fastened attention upon the integrable cases, or at any rate had been content to reduce the order as far as possible by means of known integrals. In the field of celestial mechanics, Laplace and succeeding theoretical astronomers had been content to use formal series as a means of systematic computation. But Poincaré was the first to attack the general questions of *non-integrable* systems of differential equations from a purely mathematical point of view. His principal effort in this direction for $n=2$ is contained in the memoir just cited. Most of his later work in celestial mechanics may be regarded as concerned with special important cases where $n>2$, in particular with the restricted problem of three bodies for which n reduces to 3.

Even in the case $n=2$, interesting problems of analysis situs presented themselves to Poincaré. One of these of much importance was the following:† Suppose that there be given a one-to-one direct continuous transformation of the circumference of a circle into itself, namely $\theta_1=f(\theta)$, where θ is the angular coordinate of a point P, and θ_1 is the coordinate of the transformed point P_1. Poincaré proved that there always exists a unique number σ such that the nth iterate P_n of P by T has a coordinate θ_n intermediate in value between

$$n\sigma - 2\pi \quad \text{and} \quad n\sigma + 2\pi, \qquad (n = 1, 2, 3, \cdots).$$

* *The later scientific work of Henri Poincaré*, Rice Institute Pamphlets, vol. 20 (1933).

† See Chapter 15 of the third part of his memoir.

In other words, the constant σ measures the mean rotation of the point P under indefinite iteration of T. For this reason σ may be called the "coefficient of rotation" of T. Poincaré also proved that if $\sigma/(2\pi)$ is rational there exist periodic points P such that $P_n=P$; in this case the structure of T may be regarded an entirely known. Furthermore, if $\sigma/(2\pi)$ is irrational, *and if every point of the circumference is taken into the vicinity of every other point by successive iteration,* he showed that the transformation T is topologically equivalent to a pure rotation: $\phi_1=\phi+\sigma$. But Poincaré was not able to prove that the italicized hypothesis above is always satisfied even in the case when $f(\theta)$ is analytic, although if $f(\theta)$ is restricted to be merely continuous he was able to prove that the hypothesis need not be satisfied. This highly difficult and interesting open question was brilliantly solved by Denjoy in 1932,* who proved that if $f(\theta)$ is continuous and of limited total variation, the hypothesis is automatically fulfilled.

It was Poincaré's attempt to deal with the case $n>3$ which led him to his later work on analysis situs, for he himself says in his *Analyse*: "In order to go further it was necessary for me to create a geometric instrument which was lacking when I wished to penetrate into space of more than three dimensions. This was the principal reason which decided me to undertake the study of analysis situs." Furthermore, in connection with the above specified problem solved by Denjoy, Poincaré remarks, in effect, that the analogy between some of his results and known facts in celestial mechanics would cause him without doubt to return to the problem later. Thus his studies in this memoir led him directly to his subsequent fundamental work in the fields of analysis situs and celestial mechanics.

Poincaré began his study of non-linear ordinary differential equations in the cases $n=2$, 3 referred to above, and his study of ordinary linear differential and difference equations of order n almost simultaneously. In fact although his first paper on linear difference and differential equations appeared in the American Journal of Mathematics in 1883, most of the results had been contained in his unsuccessful Prize Essay of 1880, as he himself states. The work of Fuchs and others on ordinary linear differential equations had been limited mainly to the case of regular singular points—a highly special case—while Poincaré dealt with irregular singular points as well. In this connection it is necessary to refer to a very illuminating earlier paper by the physicist G. G. Stokes published in 1857, entitled *On the discontinuity of arbitrary constants which appear in divergent developments.*† Here Stokes examined in detail the behavior of the solutions of Bessel's equation in the whole complex plane; this equation has an irregular singular point at infinity. Poincaré does not seem to have known of this highly suggestive paper since he nowhere refers to it.

The main advance of Poincaré in this domain was to show that, under certain conditions, the known formal series solutions of Thomé and Fabry represented actual solutions of the linear differential equation in suitable sectors of the complex plane in the same asymptotic sense in which Stirling's formula was known to represent the function $\Gamma(n)$. An analogous result was found by

---

* *Sur les caractéristiques à la surface du tore*, Comptes Rendus, vol. 194 (1932).

† Transactions of the Cambridge Philosophical Society, vol. 10.

him for linear difference equations. The papers by Poincaré in these fields went a long way toward providing a general theory of such linear equations. However, it is only very recently that a general theory of such equations has been developed.*

The last paper of the volume under review contains the second part of the unsuccessful Prize Essay referred to above, and was only published posthumously in the Acta Mathematica in 1923. Here we find a preliminary treatment of the problem as to when an ordinary linear differential equation of the second order with polynomial coefficients and only regular singular points is such that if we write $y_1(x)/y_2(x)=z$ when y_1 and y_2 are two linearly independent solutions, then x is a meromorphic function of z. This problem had been proposed and incompletely solved by Fuchs in 1880, and Poincaré's essay is mainly a critique of Fuchs's work. In this article is to be found the genesis of Poincaré's work in the theory of automorphic functions, which is contained in volume II of his Collected Works.

The reader of volume I will also be very grateful for the careful and highly competent revision which Professor Drach has provided.

G. D. Birkhoff

EISENHART ON CONTINUOUS GROUPS

Continuous Groups of Transformations. By L. P. Eisenhart. Princeton, Princeton University Press, and London, Oxford University Press, 1933. 10+299 pp.

This work is an up-to-date textbook on Lie's theory of continuous groups and on the recent developments of this theory. It is, in contrast to some introductions to the subject, not a book written for a beginner who knows only calculus. It will, however, be enjoyed by every student who knows some existence theorems on differential equations and is familiar with the elements of the tensor symbolism. He will not be bored through a considerable part of the book by the almost traditional detailed treatment of more or less trivial examples before reaching the first general theorem of Lie. In fact, in the present book many illustrations of the general theory are formulated only as exercises, so that Lie's three Fundamental Theorems can be developed fully in the very first chapter. Due to a similar attitude through the whole work, the present book, though of moderate extent, leads essentially farther and deeper than its predecessors, without, however, requiring too much from the reader.

The book starts out with some elementary facts regarding total and Jacobian differential systems and their generalizations. The necessary existence and uniqueness theorems of local character are used only under the restriction of analyticity. These preparatory paragraphs are followed by an explanation of the notion of a continuous transformation group and by a rather concise

* See a joint paper by myself and Trjitzinsky, *Analytic theory of linear difference equations*, Acta Mathematica, vol. 60 (1932), and a paper by Trjitzinsky entitled *Analytic theory of linear differential equations*, in vol. 62 of the same journal.