and between the points \(m/2^n\), let \(f_n\) be defined linearly.

6. A Set of Conditions Sufficient to Insure Convergence in Variation.

**Theorem 7.** Let \(f_n(x)\) be a sequence of absolutely continuous functions converging to a limit function \(f_0(x)\) on \((a, b)\); let \(f'_n(x)\) converge asymptotically to a limit function, and let \(f'_n(x)\), \((n = 1, 2, 3, \cdots)\), be dominated by a summable function; then we have \(f_n(x) \rightarrow f_0(x)\) on \((a, b)\).

It is easily seen that the hypotheses imply (i) that \(f_0(x)\) is absolutely continuous, so that we may write

\[
T_a^b f_n(x) = \int_a^b |f'_n(x)| \, dx, \quad (n = 0, 1, 2, \cdots),
\]

and (ii) that we may pass to the limit under the integral sign.

**Corollary.** Let the series \(\sum_{i=0}^{\infty} a_i x^i\), with real coefficients, have the radius of convergence \(R(>0)\); let the sum of the series be denoted by \(S(x)\), and let \(S_n(x) = \sum_{i=0}^{n} a_i x^i\); then we have \(S_n(x) \rightarrow S(x)\) on each interval \((a, b)\), \((-R < a < b < R)\).

**Types of Involutorial Space Transformations Associated with Certain Rational Curves—Composite Basis Curves**

**By Amos Black**

1. **Introduction.** In a preceding paper† the author found and discussed the involutoral transformations belonging to the special complex of lines which meet a rational curve \(r\) of order \(m\), \((m = 2, 3, 4, 5)\), and having a pencil of invariant cubic surfaces which contain the curve \(r\) as a simple basis element, with the restriction that the residual basis curve, \(\gamma_{9-m}\), of the pencil should not be composite. In this paper we shall discuss the cases where \(\gamma_{9-m}\) is composite.

2. **Equations of the Transformation.** The equations of the

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* Presented to the Society, April 14, 1933.
† *Types of involutoral space transformations associated with certain rational curves*, Transactions of this Society, vol. 34 (1932), pp. 795–810.
transformation and the image of \( r_m \) are identical with those of the preceding paper (p. 797), namely:

\[
x_t = y_t R - z_t M, \quad (i = 1, 2, 3, 4),
\]

and

\[
I_{6m+6}:
\begin{align*}
M_{3m+6} :& \quad (m+1) + 2t \quad m+1 \\
r_m & \sim R_{6m+4} : \quad (2m+1) + 2t \quad 2m+1 \\
K_{(m+1)(3m-6)} : & \quad r_m \quad \gamma_{9-m},
\end{align*}
\]

where \( \gamma_{9-m} \) may or may not be composite.

Let \( \gamma_{9-m} \) be composite, and let one part be \( \gamma_n \) (1 < \( n < 9 - m \)). Fix a point \( P \) on \( \gamma_n \). Let \( L \) be an arbitrary point on \( r_m \). The line \( PL \) meets \( F_L \) in \( P, L, \) and a third point \( P' \), the image of \( P \). As \( L \) describes \( r_m \), the line \( PL \) generates a cone \( K_m \) with one point \( P' \) on each generator. Then the locus of \( P' \) is a curve of order \( m+1 \) the number of times \( PL \) is tangent to \( F_L \) at \( P \). Given a point \( L \), the tangent plane of \( F_L \) at \( P \) intersects \( r_m \) in \( m \) points \( K \). Conversely, given a point \( K \), the \( F_3 \) whose tangent plane at \( P \) passes through \( K \) is unique; hence there is only one point \( L \). This \((1, m)\) correspondence on \( r_m \) has \( m+1 \) coincidences, and

\[
P \sim C_{2m+1} : \quad P^{m+1}.
\]

As \( P \) traces \( \gamma_n \), the \( C_{2m+1} \) generates a surface \( \Gamma \) which is the image of \( \gamma_n \).

The surface \( \Gamma \) may be found in an alternate manner. Let \( O \) be a fixed point on \( r_m \) and \( P \) an arbitrary point on \( \gamma_n \). The line \( PO \) cuts \( F_0 \) in \( O, P, \) and a third point \( P' \) which is the part of the image of \( P \) which lies on \( F_0 \). As \( P \) traces \( \gamma_n \), \( PO \) generates a cone \( K_n \), and \( P' \) generates a curve \( C \) which is the part of the image of \( \gamma_n \) which lies on \( F_0 \). The cone \( K_n \) and \( F_0 \) intersect in \( \gamma_n \) and \( C \). The surface \( \Gamma \) is obtained by eliminating the parameter \((\lambda, \mu)\) between \( K_n \) and \( F_0 \).

Let us consider a cone \( K_n \). It stands on \( \gamma_n \) and has its vertex at an arbitrary point \( O (\lambda, \mu) \) on \( r_m \), and is met by \( r_m \) in \( n(m-1) - i \) arbitrary points other than \( O \), where \( i \) is the number of intersections of \( r_m \) and \( \gamma_n \). Then \( K_n \) is met by \( r_m \) in one \( n \)-fold and \( n(m-1) - i \) simple points, hence the parameter \((\lambda, \mu)\) enters the equation of \( K_n \) to degree \( n + n(m-1) - i = mn - i \). If the parameter is eliminated between \( K_n (x, \lambda, \mu) : \gamma_n \), and
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\[ F_0 = \mu F_3 - \lambda F_3' : \quad r_m \gamma_n \gamma_{9-m-n}, \]

we have

\[ \Gamma(\alpha, F_3, F_3' ) \]

of degree \( n \) in \( (x) \), and of degree \( mn - i \) in \( F_0, F_3' \). Hence

\[ \Gamma_{3(mn-i)n} : \quad r_m \gamma_n \gamma_{9-m-n}. \]

Since \( C_{2m+1} : P^{m+1} \) is perspective from \( P \), then \( P \) is invariant in the \( m+1 \) directions of the tangents of \( C_{2m+1} \) at \( P \). Thus \( m+1 \) sheets of \( \Gamma \) touch the \( m+1 \) sheets of \( M \) along \( \gamma_n \).

From the table of images and intersections we find that \( n \) sheets of \( \Gamma \) have a common tangent plane at all points \( O \) of \( r_m \), the tangent plane of \( F_0 \) at \( O \).

There is one set of transformations which involves additional explanation; namely, that where \( m = 2 \) and the residual \( \gamma_6 \) is composite, consisting of a straight line which meets \( r_2 \) twice and a sextic where the sextic may, or may not, be composite. We shall treat the case where the sextic is not composite.

3. Equations and Images. The fundamental curves are \( r_2, \gamma_1, \gamma_6, (\phi = 4) \), where \( [r_2, \gamma_1] = 2 \) points, \( [r_2, \gamma_6] = 4 \) points, and \( [\gamma_1, \gamma_6] = 2 \) points. We see that \( \gamma_1, r_2 \) forms a complete plane section of any \( F_3 \). If we choose \( P \) as a general point of this plane, the particular \( F_0 \) determined by \( P \) is composite and has the plane for one component. Thus the whole line \( PO \) lies on \( F_3 \), hence is parasitic. But there is a pencil of such lines in the plane and the plane factors out of the transformation. The degree of the transformation is reduced by one, hence is sixteen. The plane is also a factor of \( R \) and \( M \).

A plane \( \sim S_{16} : \quad r_2 \gamma_1 \gamma_6, \quad r_2 \gamma_1 \gamma_6, \quad r_2 \gamma_1 \gamma_6, \)

Fix a point \( O \) on \( r_2 \). The line \( OL, \) where \( L \) is an arbitrary point on \( r_2 \), meets \( F_L \) in a point \( P' \) on \( \gamma_1 \). As \( L \) describes \( r_2, P' \) generates \( \gamma_1 \) which is the image of \( O \). But as \( O \) describes \( r_2, \gamma_1 \) remains fixed, hence there is no surface \( R' \).

The line joining a fixed point \( P \) on \( \gamma_1 \) to an arbitrary point \( O \)
on \( r_2 \) meets \( F_0 \) in a point \( P' \) on \( r_2 \). As \( O \) describes \( r_2 \), \( P' \) also describes \( r_2 \). However, when \( O \) is either of the points of intersection of \( \gamma_1, r_2 \), the point \( P' \) can not only be a point of \( r_2 \) but also any point on \( \gamma_1 \). Thus the image of any point \( P \) on \( \gamma_1 \) is \( r_2 \), and \( \gamma_1 \) counted twice. But as \( P \) traces \( \gamma_1 \), the image of \( P \) remains fixed; hence there is no surface which is the image of \( \gamma_1 \).

The two curves \( r_2, \gamma_1 \) each play a dual role. The curve \( r_2 \) is a fundamental curve of the first species with image \( R_{35} \), and a fundamental curve of the second species with image \( \gamma_1 \). The line \( \gamma_1 \) is a fundamental line of the second species with image \( r_2 \) and a parasitic line.

The reduction in the degree of the transformation does not affect the image of \( \gamma_6 \); hence

\[
\gamma_6 \sim \Gamma_{35} \quad r_2 \quad \gamma_1 \Gamma_6.
\]

The Jacobian is \( J_{35} = R_{35} \Gamma_{35} \).

4. Number of Parasitic Lines. We have already seen that \( \gamma_1 \) is itself a parasitic line counted twice. We shall think of it as two parasitic lines. In a manner similar to that of the preceding paper (pp. 798–800) we find twenty-two other parasitic lines each of which meets \( r_2 \) once and \( \gamma_6 \) twice. In all there are twenty-four parasitic lines, which is five less than the number in the case for \( \gamma_7 \) non-composite.

The procedure in this and the preceding paper can be immediately generalized to spaces of higher dimensions. One interesting feature of the generalization is that images of lines, planes, three-way spaces, etc., must all be considered. No essentially new ideas are introduced.

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