ON THE FINITENESS OF THE CLASS NUMBER IN A SEMI-SIMPLE ALGEBRA*

BY C. G. LATIMER

1. *Introduction.* Let A be a rational semi-simple algebra of order n and let G be a domain of integrity of order n in A, according to Dickson's definition.† In a recent paper,‡ Miss Shover showed that if A is a division algebra, the number of classes of left ideals in G is finite. She used the definitions of an ideal and a class of ideals as given by MacDuffee.§ We shall extend this result to any A. Since Miss Shover also showed that there is a one-to-one correspondence between the classes of left ideals and the classes of right ideals in G, it will be sufficient to prove the theorem for right ideals. By applying this result, we shall obtain a theorem on similar matrices.

Artin proved the finiteness of the right ideal class number for a maximal ordnung in A, using a different definition of an ideal and a class of ideals.‖ Every domain of integrity of order n is an ordnung. An ordnung is a domain of integrity of order n if and only if it contains the modulus of A. In particular, every maximal ordnung is a domain of integrity of order n. Every non-singular ideal according to MacDuffee is an ideal according to Artin and if β is an element and K is an ideal in G, MacDuffee's and Artin's definitions of their norms $N(\beta)$, $N(K)$ are the same.

2. *Proof of the Theorem.* The only place in her paper where Miss Shover employed the hypothesis that A is a division algebra was in obtaining, for left ideals, a result equivalent to the following, which was proved by Artin for the case where G is maximal.

LEMMA 1. *There is a positive number C, depending only on G, such that if K is a non-singular right ideal in G, there is an element β in K for which*

* Presented to the Society, December 27, 1933.

† *Algebren und ihre Zahlentheorie*, p. 155.

‡ This Bulletin, vol. 39 (1933), pp. 610–14.

§ Transactions of this Society, vol. 31 (1929), pp. 71–90.

‖ Abhandlungen, Mathematisches Seminar, Hamburg, vol. 5 (1927), pp. 251–289.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$0 < |N(\beta)| \leqq CN(K).$$

There is a maximal domain G_0 containing G.* Consider the following lemma.

LEMMA 2. *Lemma* 1 *is valid if it is valid when G is replaced by G_0.*

This lemma was proved by Artin, using a different notation, for the case where G and G_0 are both maximal.† However, the present lemma may be proved by exactly the same argument as that made by Artin.

But by Artin's Theorem 17 (p. 283), our Lemma 1 is valid for G_0. Lemma 1 follows.

Let K be a non-singular right ideal in G and let β be an element in K satisfying the conditions of Lemma 1. Following Miss Shover's proof, we find that the transpose of the first matrix of β is $R(\beta)=MG$, where G is the matrix of K and M is a matrix with integral elements. Noting that $|R(\beta)|=N(\beta)\neq 0$ and letting $\pm|M|=m>0$, we have

$$0 < |N(\beta)| = \pm|M| \cdot |G| = m\cdot N(K) \leqq C\cdot N(K).$$

Furthermore, the adjoint of M is the matrix of an ideal L, in the same class as K. Then by the last of the above inequalities, we have

$$0 < N(L) = m^{n-1} \leqq C^{n-1}.$$

Hence every class of ideals contains a non-singular ideal with norm $\leqq C^{n-1}$. Since there is only a finite number of ideals with norms equal to a given positive integer, we have the following theorem.

THEOREM 1. *The number of classes of right ideals in G is finite.*

3. *An Application of Theorem* 1. It will be understood that all matrices referred to are square matrices of order n with integral elements. Two matrices, A and A_1, are said to be similar if there is a unimodular matrix Z, such that $A_1=ZAZ^{-1}$. All matrices similar to the same matrix are said to form a class. Two

* Artin, loc. cit., p. 265.

† Loc. cit., pp. 283–284.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

matrices are similar if and only if they belong to the same class. Let

$$f(x) = x^n + k_1x^{n-1} + \cdots + k_n,$$

where the k's are rational integers, $k_n \neq 0$, and $f(x)=0$ has no multiple roots. If A is a matric root of $f(x)=0$ and is non-derogatory, that is, is not a root of an equation, with rational coefficients, of lower degree, the same is true of every matrix similar to A. It is known that there is a one-to-one correspondence between the classes of ideals in a domain of integrity in a certain commutative semi-simple algebra and the classes of non-derogatory matrices which are roots of $f(x)=0$.* We have therefore, by Theorem 1, the following result.

THEOREM 2. *The number of classes of non-derogatory similar matrices which are roots of* $f(x)=0$ *is finite.*

THE UNIVERSITY OF KENTUCKY

KASNER'S CONVEX CURVES†

BY MORRIS HALPERIN

1. *Preliminary Discussion.* A Kasner convex curve is the limit of a sequence of simple, closed, convex polygons, $P_0, \cdots, P_n, \cdots$, each of which has a finite number of sides and is obtained from the preceding one by measuring off the rth part of the length of each side from both its ends and cutting off the corners. The number r is restricted to the inequality $0<r<1/2$. To obtain an analytic definition for the curve, we proceed as follows. We note that the centroid of the vertices of P_0 is also the centroid of the vertices of every P_n. Hence G is interior to every P_n. Let $z_n(t)$ be the intersection of a ray from G of inclination t with the polygon P_n. The sequence of functions $\{z_n(t)\}$ will be found to converge uniformly to a function $z(t)$.

* Latimer and MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Annals of Mathematics, (2), vol. 34 (1933), pp. 313–316.

† Presented to the Society, February 25, 1933. Another paper will follow in which additional properties of these curves will be discussed; particularly their second derivatives, their non-analytic character, and their areas. See this Bulletin, Abstract 39-3-68.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use