

METRICALLY TRANSITIVE POINT TRANSFORMATIONS*

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1. *Introduction.* Let

$$(1) \quad T^{(j)}: \quad \xi_i^{(j)} = \xi_i^{(j)}(x_1, x_2, \dots, x_n),$$

$$(i = 1, 2, \dots, n; j = 1, 2, \dots),$$

denote a denumerable set of point transformations in n -dimensional euclidean space, the transformations of which leave a certain p -dimensional ($p \leq n$) region R (region \equiv open, connected point set) of this space invariant. The case where R is the entire n -dimensional space is not excluded.

The set of transformations (1) will be said to be metrically transitive with respect to R , if the complement set (with respect to R) of every non-zero subset of R that is invariant under each transformation of the set (1) is a zero set.† If the transformations of the set (1) form a group, this group is said to be metrically transitive with respect to R .

If the set (1) consists of a single transformation T and if $p = 2$, the above definition is sensibly the same as that given by Birkhoff and Smith‡ for metrically transitive surface transformations. If the transformations of the set (1) are composed of the iterations of a single transformation T and its inverse T^{-1} , together with the identity transformation, the transformations of the set form an infinite cyclic group which is metrically transitive with respect to a certain region if, and only if, one of the elements of this group, exclusive of the identity transformation, is metrically transitive with respect to this region.

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† *Non-zero set* \equiv a set of positive p -dimensional Lebesgue measure. *Zero set* \equiv a set of zero p -dimensional Lebesgue measure. In this paper the word measure refers to Lebesgue measure; the measure of a measurable set S is denoted by mS .

‡ G. D. Birkhoff and P. A. Smith, *Structure analysis of surface transformations*, Journal de Mathématiques, vol. 7 (1928), p. 365.

The principal result of this paper is that the elliptic modular group

$$(2) \quad \zeta = \frac{\alpha z + \beta}{\gamma z + \delta},$$

(where α , β , γ , and δ are integers for which $\alpha\delta - \beta\gamma = 1$ and ζ , z complex variables) is metrically transitive with respect to the real axis. Here $n=2$, $p=1$, and R is the real axis. From the metrical transitivity of this group there follow several results in regard to: (a) the covering of any given point set on the real axis by a denumerable set of point sets,* (b) the distribution of equivalent points [that is, points which may be transformed into one another by transformations of the group (2)] on the real axis, (c) the measure of the sets of Liouville and Hurwitz† numbers on the real axis. In conclusion we consider a very simple metrically transitive set of point transformations and point out an application of our results to a metrically transitive system previously considered by Birkhoff and Smith,‡ and others.

2. *Metrical Transitivity With Respect to the Real Axis.* In order to demonstrate that the group (2) is metrically transitive with respect to the real axis, it is convenient to consider directly the group of transformations

$$(3) \quad \xi = \frac{\alpha x + \beta}{\gamma x + \delta},$$

(where α , β , γ , and δ are integers for which $\alpha\delta - \beta\gamma = 1$ and ξ , x are real variables) of the real axis into itself, induced on it by the group (2). Let S be any non-zero set on the x -axis invariant under all the transformations of the group (3). Since this group contains the translations $\xi = x \pm n$, ($n=0, 1, \dots$), the comple-

* For other results in this connection see H. Rademacher, *Eineindeutige Abbildungen und Messbarkeit*, Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 184–195.

† A number is said to be a Hurwitz number if its development into a regular continued fraction is a Hurwitz continued fraction. See O. Perron, *Die Lehre von den Kettenbrüchen*, 1913, pp. 126–127.

‡ Loc. cit., p. 368. See also G. D. Birkhoff, *Probability and physical systems*, this Bulletin, vol. 38 (1932), p. 371.

mentary set of S on the x -axis may be shown to be a zero set by demonstrating that the part S_1 of S contained in the unit interval $0 < x < 1$ necessarily has measure 1; and it is to the proof of this fact that we now turn.

3. *Measure of S_1 on the Unit Interval.* Let ξ_1 be any irrational number in the unit interval $0 < \xi < 1$ and denote by $[0, p_1, p_2, \dots, p_j, \dots]$ its development into an unending regular continued fraction.* We consider the subintervals $I^{(n)}$ of the unit interval $0 < \xi < 1$ defined by

$$(4) \quad I^{(n)}: [0, p_1, p_2, \dots, p_{2n}] < \xi < [0, p_1, p_2, \dots, p_{2n}, 1],$$

$$(n = 1, 2, \dots),$$

each of which contains the point ξ_1 . From the theory of continued fractions it is known that these intervals are the transforms of the unit interval $0 < x < 1$ by the transformations $\xi = [0, p_1, p_2, \dots, p_{2n}, 1:x]$, or

$$(5) \quad T^{(n)}: \xi = \frac{P_{2n}x + P_{2n-1}}{Q_{2n}x + Q_{2n-1}} = \frac{P_{2n-1}x + P_{2n}}{Q_{2n-1}x + Q_{2n}}, \quad (n = 1, 2, \dots),$$

and hence are identical with the intervals

$$(4)' \quad I^{(n)}: \frac{P_{2n}}{Q_{2n}} < \xi < \frac{P_{2n-1} + P_{2n}}{Q_{2n-1} + Q_{2n}}, \quad (n = 1, 2, \dots).$$

Here $P_{2n-1}, P_{2n}, Q_{2n-1}$, and Q_{2n} are positive integers calculated from the recursion formulas

$$P_0 = 0, P_1 = 1, \dots, P_j = p_j P_{j-1} + P_{j-2},$$

$$Q_0 = 1, Q_1 = p_1, \dots, Q_j = p_j Q_{j-1} + Q_{j-2}, \quad (j = 2, 3, \dots, 2n).$$

They satisfy the following inequalities:

$$(6) \quad (a) P_{2n-1}Q_{2n} - P_{2n}Q_{2n-1} = 1; \quad (b) Q_{2n} \geq 2n; \quad (c) Q_{2n-2} < Q_{2n};$$

$$(n = 1, 2, \dots).$$

Hence the lengths $mI^{(n)}$ of the intervals $I^{(n)}$ are given by

$$(7) \quad mI^{(n)} = \frac{P_{2n-1} + P_{2n}}{Q_{2n-1} + Q_{2n}} - \frac{P_{2n}}{Q_{2n}} = \frac{1}{Q_{2n}(Q_{2n-1} + Q_{2n})} < \frac{1}{2n}.$$

* For the notation and properties of continued fractions used in this paper, see O. Perron, op. cit.

From (6a) it follows that the transformations $T^{(n)}$ in (5) belong to the group (3). Since the transformation $T^{(n)}$ transforms the unit interval $0 < x < 1$ into the interval $I^{(n)}$, and, since ξ_1 is contained in each of the intervals $I^{(n)}$, it follows from (7) that the unit interval $0 < x < 1$ is transformed by the transformations $T^{(n)}$ into a sequence $\{I^{(n)}\}$ of intervals closing down on ξ_1 .

We observe that the transformation $T^{(n)}$ can be written in the form

$$(8) \quad T^{(n)}: \quad \xi = \frac{P_{2n}}{Q_{2n}} + \int_0^x \phi_n(x) dx, \quad (n = 1, 2, \dots),$$

where

$$(9) \quad \phi_n(x) = (Q_{2n-1}x + Q_{2n})^{-2} > 0, \quad (0 \leq x \leq 1; n = 1, 2, \dots).$$

From (6a) we conclude that $\phi_n(x)$ is continuous in this interval and hence summable in any subinterval of it. The transformation $T^{(n)}$ is therefore an absolutely continuous function of x in the interval $0 < x < 1$ and any zero set in the interval $0 < x < 1$ is accordingly transformed by it into a zero set* in the interval $I^{(n)}$.

The set S_1 (see §2) is contained in a set \bar{S}_1 which is the inner limiting set of a monotone decreasing sequence of sets of non-overlapping intervals, such that $m\bar{S}_1 = mS_1$. If we denote by $\Sigma_1^{(n)}$ and $\bar{\Sigma}_1^{(n)}$ the sets in the interval $I^{(n)}$ into which $T^{(n)}$ transforms S_1 and \bar{S}_1 , respectively, we have†

$$(10) \quad m\Sigma_1^{(n)} = m\bar{\Sigma}_1^{(n)} = \int_{\bar{S}_1} \phi_n(x) dx = \int_0^1 f(x)\phi_n(x) dx,$$

where $f(x)$ is the characteristic function of the set \bar{S}_1 . Since $\phi_n(x)$ is continuous and $f(x)$ a summable function that is ≥ 0 in the interval $0 \leq x \leq 1$, we may apply the first mean value theorem to obtain

$$(11) \quad \int_0^1 f(x)\phi_n(x) dx = \phi_n(x_1) \int_0^1 f(x) dx = \phi_n(x_1)mS_1,$$

* See, for example, C. Carathéodory, *Vorlesungen über Reelle Funktionen*, 1927, p. 583.

† See, for example, E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 1921, p. 593.

($0 \leq x_1 \leq 1$). Now the set S is invariant with respect to the group (3) and hence invariant with respect to the transformations $T^{(n)}$. Accordingly $\Sigma_1^{(n)} \subset S_1$ and the density of S_1 in the interval $I^{(n)}$ is therefore not less than the density of $\Sigma_1^{(n)}$ in this interval. From (7), (9), (10) and (11) we have then

$$(12) \quad \frac{mI^{(n)} \cdot S_1}{mI^{(n)}} \geq \frac{m\Sigma_1^{(n)}}{mI^{(n)}} = \frac{Q_{2n}(Q_{2n-1} + Q_{2n})}{(Q_{2n-1}x_1 + Q_{2n})^2} mS_1 = \frac{1 + \theta}{(1 + \theta x_1)^2} mS_1,$$

where $\theta = Q_{2n-1}/Q_{2n}$. From (6c) we obtain $0 < \theta < 1$, and therefore from (12), and our hypothesis $mS_1 > 0$, there results

$$(13) \quad \lim_{n \rightarrow \infty} \frac{mI^{(n)} \cdot S_1}{mI^{(n)}} \geq \lim_{n \rightarrow \infty} \frac{m\Sigma_1^{(n)}}{mI^{(n)}} > 0.$$

From (13) it follows that, if the metrical density of S_1 exists at the point ξ_1 , it is positive. Since ξ_1 is any irrational point of the interval $0 < \xi < 1$, the metrical density of S_1 is either positive or non-existent on the set of irrational points in this interval. However, from the theorem of Lebesgue† on the metrical density of measurable sets, we know that the metrical density of S_1 exists, and is equal to unity at all points of S_1 , with the possible exception of a component which is a zero set, and is zero at all points of the complement of S_1 with respect to the interval $0 < \xi < 1$, with the possible exception of the points of a component which is a zero set. Therefore $mS_1 = 1$, which was to be proved.

4. *Conclusions.* From the metrical transitivity of the group (2) with respect to the real axis we are able to draw several obvious conclusions. The first result is that every point set on the real axis may be covered, with the possible exception of the points of a component which is a zero set, by a denumerable set of linear sets obtained from any previously given point set of positive measure on the real axis by means of the transformations of the group (2). In order to demonstrate this let S_2 be any set of positive measure on the real axis and denote by S_i , ($i=1, 2, \dots$), the sets obtained from S_2 by the transformations of the group (2). The set $M = \sum_{i=1}^{\infty} S_i$ is obviously invariant under the transformations of the group (2). Since it is of positive measure,

† See, for example, E. W. Hobson. op. cit., p. 181.

and the group (2) is metrically transitive with respect to the real axis, its complementary set is a zero set. Consequently M must cover any set up to a component of measure zero. This last result is very closely related to a result of Rademacher* obtained by use of the covering theorem of Vitali.

Another result is that any set of positive measure on the real axis necessarily contains equivalent points. For, if a set of positive measure exists which contains no equivalent points, it may be divided into two sets, each of positive measure, such that no point of one is equivalent to any point of the other. A set M , constructed from one of these sets as was the set M above, would, however, cover the other up to the points of a zero set, thus leading to a contradiction.

Finally, we observe that, since the sets of Liouville and Hurwitz numbers on the real axis are invariant† under the transformations of the group (2), either they, or their complementary sets with respect to the real axis, are zero sets.

5. *An Example.* A simple example of a set of point transformations which is metrically transitive with respect to a region R is obtained when the transformations of the set (1) are the translations

$$(14) \quad T^{(i)}: \quad \xi = x + a_j, \quad (j = 1, 2, \dots),$$

for which

$$(15) \quad \lim_{n \rightarrow \infty} a_j = 0.$$

Here $n = p = 1$ and R is the x -axis (that is, the entire 1-dimensional euclidean space). In order to demonstrate the metrical transitivity of (14) with respect to the x -axis, let S be any measurable set invariant under all the transformations of the set (14). Because of (15), the set S is certainly invariant under the transformations of a subset of (14), namely,

$$\xi = x + \alpha_j, \quad (j = 1, 2, \dots),$$

in which $\lim_{j \rightarrow \infty} \alpha_j = 0$ and $\alpha_i > \alpha_j$ if $i < j$. Starting from any point of the x -axis we lay off upon it a denumerable set of intervals,

* J. Rademacher, loc. cit.

† O. Perron, op. cit., p. 127 and p. 141.

each of length α_1 and possessing only their end points in common with one another, which cover the entire x -axis without lacunae. The density of the set S is obviously the same in each of these intervals and consequently the same in any finite segment of the x -axis whose end points coincide with end points of these intervals. This process is then repeated for intervals of length α_2 , of length α_3 , etc. One obtains in this manner successive subdivisions of the x -axis into intervals, the lengths of which are arbitrarily small and in each of which the set S possesses, for any given subdivision, the same density. Let I_1 and I_2 be any two intervals of the x -axis and let $I_1(\alpha_k)$ and $I_2(\alpha_k)$ denote the two subintervals of I_1 and I_2 , respectively, which are built up from all those intervals of the k th subdivision that are entirely contained in I_1 and I_2 . Obviously $\lim_{k \rightarrow \infty} mI_1(\alpha_k) = mI_1$ and $\lim_{k \rightarrow \infty} mI_2(\alpha_k) = mI_2$. Since the density of the set S is the same in the two intervals $I_1(\alpha_k)$ and $I_2(\alpha_k)$ for any value of k , it is the same in the two intervals I_1 and I_2 . Since I_1 and I_2 were arbitrary intervals, the set S is of homogeneous density and either it or its complementary set is of measure zero.*

In the metrically transitive system considered by Birkhoff and Smith it is shown† by a Fourier analysis that if θ and $\bar{\theta}$ are angular variables of period 1, the transformation

$$(16) \quad \bar{\theta} = \theta + \alpha, \quad (\alpha \text{ irrational}),$$

of the periphery of a circle into itself, is metrically transitive. The same result follows by means of the methods of this section, since (16) is metrically transitive, if the set of transformations

$$\xi = x + n\alpha - [n\alpha], \quad (n = 1, 2, \dots),$$

(where $[n\alpha]$ is the greatest integer, in absolute value, for which $|[n\alpha]| < |n\alpha|$) of the x -axis into itself, is metrically transitive with respect to the x -axis. That this is actually so is seen by putting $a_j = j\alpha - [j\alpha]$ in (14) and observing that (15) is fulfilled.

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* E. Jacobsthal and K. Knopp, *Bemerkungen über die Struktur linearer Punktmengen*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, No. 129 (1915), p. 122.

† G. D. Birkhoff, *Probability and physical systems*, loc. cit.