which is assumed to be such that it may be integrated term-by-term in the interval $0 \leqq x \leqq \pi$. Making use of the method by which (I) was obtained from (5), we find the expansion

$$\text{(III)} \qquad \phi(x) = \pi 2^{\nu-1} \sum_{n=1}^{\infty} \frac{a_n}{n^{\nu-1}} J_{\nu/2}^2\left(\frac{nx}{2}\right), \qquad (0 < x < \pi,\ \nu \geqq 1),$$

where

$$\phi(x^{1/2}) = p^{-1/2} x^{-1/2} f(x^{1/2}).$$

If the above method is applied to Neumann and Kapteyn series, well known expansions in terms of squares of Bessel functions are obtained. Expansions (I), (II), and (III) have seemingly never been published.

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NOTE CONCERNING GROUP POSTULATES*

BY RAYMOND GARVER

Let there be given a set of elements $G(a, b, c, \cdots)$ and a rule of combination, which may be called multiplication, by which any two elements, whether they be the same or different, taken in a specified order, determine a unique result which may or may not be an element of G. This system is called a group if it satisfies certain postulates; various sets of postulates have been given by different writers, and such matters as the independence of postulates and relations between sets of postulates have been pretty thoroughly covered. Most of this work was done in this country in the early part of the present century.†

It seems, however, that one interesting and rather important

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† See Pierpont, Annals of Mathematics, (2), vol. 2 (1900), p. 47; Moore, Transactions of this Society, vol. 3 (1902), pp. 485–492, vol. 5 (1904), p. 549 and vol. 6 (1905), pp. 179–180; Huntington, this Bulletin, vol. 8 (1902), pp. 296–300 and 388–91 and Transactions of this Society, vol. 4 (1903), p. 30, vol. 6 (1905), pp. 34–35 and 181–197; Dickson, Transactions of this Society, vol. 6 (1905), pp. 198–204.

question has been left unanswered, namely, the question of the independence of the following set of postulates:

I. *If a and b* are elements of G, the product ab is an element of G.*

II. *If a, b, c, ab, bc, $(ab)c$, $a(bc)$ are all elements of G, then* $(ab)c=a(bc)$.†

III. *If a and b are elements of G, there exists an element x of G such that $ax=b$.*

IV. *If a and b are elements of G, there exists an element y of G such that $ya=b$.*

This postulate set is a modification of that given by Weber.‡ Weber defined a finite group by I, II, and two further postulates, and then deduced III and IV, and the uniqueness of the elements x and y of III and IV, as theorems for finite groups. Noting that III and IV, with uniqueness, could not be so deduced from his postulates when the number of elements was infinite, he then added them to his set of postulates to define an infinite group. His procedure was perfectly natural, though it led to several redundancies. Huntington, in 1902, actually exhibited these redundancies, though he did not emphasize having done so until 1905.§ Moore, also in 1902, used the postulate system I, II, III, IV just as it is written above, apparently for the first time.|| He calls it W_1', and states explicitly, on page 489, "For W_1', the independence of the postulates is an open question." And it seems that the question has not as yet been answered.

However, Huntington, in 1902, replaced II by the following stronger postulate:

II′. *If a, b, c, ab, bc, $(ab)c$ are all elements of G, then* $(ab)c=a(bc)$.

* The symbols $a, b, \cdots$ as used in the postulates need not represent *distinct* elements of G.

† This postulate, the associative law, is sometimes written "If a, b, c are all elements of G, then $(ab)c=a(bc)$." This form is satisfactory in case II is never to be thought of apart from I.

‡ *Lehrbuch der Algebra*, vol. 2, 1896, pp. 3–4.

§ This Bulletin, vol. 8 (1902), pp. 296–300; Transactions of this Society, vol. 6 (1905), p. 182.

|| Transactions of this Society, vol. 3 (1902), pp. 485–492.

He was then* able to deduce I from II′, III, IV. I propose to show that the use of II′ is not essential, that is, that I can be deduced from II, III and IV. The proof, in fact, is simpler than Huntington's proof that I followed from II′, III, IV. It may be written down as follows:

Assume that a and b are elements of G.

(1) By IV, $\exists$ e in G such that $ea=a$.
(2) By IV, $\exists$ a' in G such that $a'a=e$.
(3) By III, $\exists$ x in G such that $a'x=b$.

We wish to show that $ab=x$.
(4) By III, $\exists$ p in G such that $ap=x$.
(5) By III, $\exists$ q in G such that $eq=p$.
(6) By III, $\exists$ r in G such that $ar=q$.
(7) By (1) and (6), $(ea)r=ar=q$.
(8) By (6) and (5), $e(ar)=eq=p$.
(9) By (7), (8) and II, $p=q$.
(10) By (5) and (9), $ep=p$.
(11) By (2) and (10), $(a'a)p=ep=p$.
(12) By (4) and (3), $a'(ap)=a'x=b$.
(13) By (11), (12) and II, $p=b$.
(14) By (4) and (13), $ab=x$.†

In the proof as I originally wrote it down, I deduced two lemmas from II, III, IV before obtaining the proof of postulate I. It is perhaps worth while noting that they can be obtained directly from II, III, IV.

LEMMA I. *There exists a unique element e, such that, for every element a, $ea=ae=a$.*

LEMMA II. *For every element a there exists a unique element a' such that $a'a=aa'=e$.*

However, the referee noted that the deduction of these Lemmas in their entirety introduced a few unnecessary steps; his suggestion that the proof be arranged in more direct form is followed above.

We may thus define a group by three postulates, II, III, and

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* This Bulletin, vol. 8 (1902), pp. 296–300.

† If the number of elements of G is finite, it is easily seen that I is a direct consequence of either III or IV.

IV. Further, no other set of three postulates chosen from I, II, III, and IV will suffice. For I, II and III, but not IV, or I, II, and IV, but not III, are satisfied by multiple groups, as defined by Clifford, which are not in general groups.* The simplest example of a system satisfying I, II, and III, but not IV, is given by the multiplication table.

	e	*a*
e	*e*	*a*
a	*e*	*a*

And II certainly does not follow from I, III, and IV. If the number of elements of G is n, we may give the multiplication table for G by means of a square array, n rows by n columns. If all the entries in this array are elements of G, and if each row, and each column, of the array contains every element of G, I, and III, and IV are satisfied. Yet if n is greater than 2, II need not hold; the simplest example is

	a	*b*	*c*
a	*a*	*b*	*c*
b	*c*	*a*	*b*
c	*b*	*c*	*a*

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* Annals of Mathematics, (2), vol. 34 (1933), pp. 865–871.