Hence by 9, \((pq')^* \rightarrow (pq')^*Z^*\) in \(T\).
Hence by 20 and 19, \((pq')^* \rightarrow \{(pq(pq'))^*(pq')^*\}\) in \(T\).
Hence by 9, \((p-3q)^* \rightarrow ((p-3pq)(pq-3p))^*\) in \(T\).

23.2. \([(p-3pq)(pq-3p)] \rightarrow (p-3q)\) in \(T\).

Proof. By 9 and 6, \((pp')^*\) in \(T\), whence by 16, \(Z^*\) in \(T\).
Hence by 19, when we replace \(p\) by \((pq')^*\) and \(q\) by \(Z^*\),
\[\{(pq')^*Z^*{(pq')^*}'\}\) in \(T\).
Hence by 9, \([(pq')^*Z^*] \rightarrow (pq')^*\) in \(T\).
Hence by 20 and 19, \{ \[(pq(pq'))^*[(pq)p']^*\] \} \rightarrow (pq')^*\) in \(T\).
Hence by 9, \([(p-3pq)(pq-3p)] \rightarrow (p-3q)\) in \(T\).

It will be noted that these proofs could be written out without
the use of the "star" notation, since \(p^*\) serves merely as an ab­
abbreviation for \(p-p'\), and \((pq')^*\) as an abbreviation for \(p-3q\).

ON INTEGRAL INVARIANTS OF NON-HOLONOMIC
DYNAMICAL SYSTEMS†

BY A. E. TAYLOR

1. Introduction. It is well known that there are certain in­
tegral invariants associated with holonomic dynamical sys­
tems. Cartan‡ demonstrated that it is possible to characterize
a Hamiltonian system by means of the relative integral in­
variant

\[\int_c \sum_i p_i dq_i - Hdt.\]

The purpose of this paper is to extend the theory to the case
of non-holonomic systems.

We shall adopt the following conventions in notation. There
are three ranges of indices, which we shall usually represent by

† Presented to the Society, June, 20, 1934. I wish to acknowledge my in­
debtedness to A. D. Michal for criticism and suggestions during the writing
of this paper.
‡ E. Cartan, Leçons sur les Invariants Intégraux, 1922, p. 13. Also W. F.
A. E. TAYLOR

Hereafter we shall not state explicitly these ranges for the indices.

2. The Equations of Motion. Consider a non-holonomic dynamical system with \( n \) degrees of freedom and \( n + k \) natural coordinates \( q_1, \ldots, q_{n+k} \), subject to the non-integrable relations

\[
j_n + v = a_v q_i + a_i', \quad i = 1, \ldots, n
\]

where the \( a's \) are continuous functions of \( q_1, \ldots, q_{n+k}, t \), possessing continuous first partial derivatives. For such a system, assumed conservative, the equations of motion are*

\[
\frac{d}{dt} \frac{\partial L}{\partial q_i} - \sum_{r=1}^{k} a_r \lambda_r, \quad (i = 1, \ldots, n),
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial q_{n+v}} = \frac{\partial L}{\partial q_{n+v}} + \lambda_r, \quad (r = 1, \ldots, k),
\]

where the \( \lambda's \) are Lagrangian multipliers, functions of the time.

It is to be observed that in case the system is holonomic all the \( \lambda's \) are zero, and Lagrange's equations hold in the usual form.

If we introduce new coordinates \( (q_j, p_j, t) \), where \( p_j = \frac{\partial L}{\partial q_j} \), and set

\[
H = \sum_{s=1}^{n+k} p_s q_s - L,
\]

then equations (2) become†

\[
\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \quad \frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i} - \sum_{r=1}^{k} a_r \lambda_r,
\]

\[
\frac{dp_{n+v}}{dt} = - \frac{\partial H}{\partial q_{n+v}} + \lambda_r,
\]

---


† For the transformation process, see Appell, *Traité de Mécanique Rationnelle*, vol. 2, 1911, p. 403.
and we find, in addition, that

\[
\frac{dH}{dt} = \sum_{r=1}^{k} a_r \lambda_r - \frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}
\]

3. Integral Invariants.* Consider a system of differential equations

\[
\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, t),
\]

where the \(X\)'s are continuous, together with all of their partial derivatives of the first order, in the neighborhood of the point \((x_1^0, \ldots, x_n^0, 0)\), and where not all the \(X\)'s vanish at this point. Then we can write the solution in the form

\[
x_i = f_i(t; x_1^0, \ldots, x_n^0).
\]

Let the initial values be made to depend upon a parameter:

\[
x_i^0 = x_i^0(\alpha), \quad (\alpha_0 \leq \alpha \leq \alpha_1),
\]

these functions being continuous, with continuous first derivatives, such that

\[
\sum_{i=1}^{n} \left( \frac{dx_i^0}{d\alpha} \right)^2 > 0.
\]

Putting these values in (7), we get equations of the form

\[
x_i = F_i(t, \alpha) = f_i(t; x_1^0(\alpha), \ldots, x_n^0(\alpha)),
\]

and these equations represent a tube of trajectories of the given system (6). If we envisage a regular curve \(C\) encircling this tube, in such manner that each trajectory cuts \(C\) once and only once, in a point for which \(t > 0\), then we may speak of \(C\) as a simple, closed circuit of the tube of trajectories.

Now let functions \(A_i(x_1, \ldots, x_n, t)\) and \(A(x_1, \ldots, x_n, t)\) be given, and consider

\[
\int \sum_{i=1}^{n} A_i dx_i + A dt = \int_{\alpha_0}^{\alpha_1} \left[ \sum_{i=1}^{n} \frac{\partial x_i}{\partial \alpha} A_i + A \frac{\partial t}{\partial \alpha} \right] d\alpha.
\]

If this integral has the same value for all such simple closed

* For other forms of definition, see Appell, loc. cit., vol. 2, p. 464.
circuits \( C \) encircling an arbitrarily chosen tube of trajectories of (6), it is called a relative integral invariant of the system (6). If, in particular, the circuit corresponds to a fixed instant \( t \), then the integral has the form

\[
\int_C \sum_{i=1}^n A_i \, dx_i = \int_{\alpha_i}^{\alpha_1} \sum_{i=1}^n A_i \frac{\partial x_i}{\partial \alpha} \, d\alpha.
\]

This we shall call a Poincaré* integral invariant, while (8) will be referred to as a Cartan invariant.

4. The Integral Invariant. Let there be given a non-holonomic system, subject to the kinematical conditions (1), and having the equations of motion (2). These two sets of equations define the \( q \)'s and \( \lambda \)'s as functions of the time. There is a \( 2(n+k) \)-parameter family of path curves of the system, represented by the functions

\[
q_s = q_s(t; q_1^0, \cdots, q_{n+k}^0; \dot{q}_1^0, \cdots, \dot{q}_{n+k}^0),
\]

where the \( 2(n+k) \) constants \((q_1^0, q_2^0)\) are given initial values.

Let these path curves be interpreted in the space of the variables \((q_1, \cdots, q_{n+k}, \dot{q}_1, \cdots, \dot{q}_{n+k}, t)\) and consider an arbitrary regular curve \( C_0 \) in the hyperplane \( t = 0 \):

\[
C_0: \quad q_s^0 = q_s^0(\alpha), \quad \dot{q}_s^0 = \dot{q}_s^0(\alpha), \quad (\alpha_0 \leq \alpha \leq \alpha_1).
\]

The tube of path curves emanating from the curve \( C_0 \) is the locus defined by the equations

\[
q_s = q_s(t, \alpha), \quad \dot{q}_s = \dot{q}_s(t, \alpha),
\]

obtained from (9) by means of the equations of \( C_0 \).

Now consider the action integral in the form

\[
J = \int_0^{t_1} L \, dt,
\]

where \( t_1 \) is a variable upper limit. When this integral is extended along a path curve on the tube, it becomes a function of \( \alpha \). Let the aggregate of values of \( t_1 \) correspond to a simple closed circuit \( C \) of the tube. We introduce a new variable \( u \), so that if

† It is sufficient to require that these functions be continuous, admitting continuous first derivatives, not all of which vanish simultaneously.
(12a) \[ t = t(u, \alpha), \]

then

(12b) \[ t(u_0, \alpha) = 0, \quad t(u_1, \alpha) = t_1, \quad \frac{dt}{du} = \frac{\partial t(u, \alpha)}{\partial u} = \rho > 0, \]

for all values of \( \alpha \), where \( u_1 \) and \( u_0 \) are constants. This can be done in a variety of ways, so that \( \rho \) may be taken arbitrarily,* within certain limits.

The integral (11) now becomes

(13) \[ J(\alpha) = \int_{u_0}^{u_1} Fdu, \]

where

\[ F(q_1, \cdots, q_{n+k}; q'_1, \cdots, q'_{n+k}, t') = L\left(q_1, \cdots, q_{n+k}; \frac{q'_1}{t'}, \cdots, \frac{q'_{n+k}}{t'}, t\right)t', \]

\[ t' = \frac{dt}{du}, \quad q'_s = \frac{\partial q_s}{\partial u} = \dot{q}_s. \]

We next compute† \( J'(\alpha) \), using Leibniz's rule, which is permissible since we have all the desired continuity. Upon carrying out the differentiation, and integrating by parts, we find

\[
J'(\alpha) = \int_{u_0}^{u_1} \rho \left[ \sum_{s=1}^{n+k} \left(\frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right) \frac{\partial q_s}{\partial \alpha} \right] \, du \\
+ \left\{ \frac{\partial L}{\partial t} - \frac{d}{dt} \left[ L - \sum_{s=1}^{n+k} \frac{\partial L}{\partial \dot{q}_s} \dot{q}_s \right] \right\} du \\
+ \left[ \sum_{s=1}^{n+k} \frac{\partial L}{\partial \dot{q}_s} \frac{\partial q_s}{\partial \alpha} + \left( L - \sum_{s=1}^{n+k} \frac{\partial L}{\partial \dot{q}_s} \dot{q}_s \right) \frac{\partial t}{\partial \alpha} \right]_{u_0}^{u_1}.
\]

Using (2), (3), and (5), we obtain from this

* It is sufficient to require that \( \rho = \rho(t, \alpha) \) be a positive integrable function of \( t \) for every \( \alpha \).
† The prime denotes differentiation with respect to \( \alpha \).
Let us now integrate \( J'(\alpha) \) around the curve \( C \) determined by
\[
\frac{\partial q_i}{\partial \alpha} + \frac{\partial t}{\partial \alpha} - \frac{\partial q_{i+n}}{\partial \alpha} \right) \lambda_r \right] du
\]

(14)

\[
+ \left[ \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] u_1 \]

The integral on the right is taken around the curve \( C_0 \), and that on the left around the simple closed circuit \( C \). Since the right side is independent of the circuit \( C \), the expression on the left of (15) is an integral invariant of the Cartan type, as defined in §3. In case the system is holonomic, we get the usual Cartan invariant (see Introduction). Equation (15) includes as a special case the Poincaré invariant
\[
\int_{\alpha_0}^{\alpha_1} d\alpha \int_{u_0}^{u_1} \rho \left[ \sum_{r=1}^{k} \left\{ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial \alpha} + a_r \frac{\partial t}{\partial \alpha} - \frac{\partial q_{i+n}}{\partial \alpha} \right\} \lambda_r \right] du
\]

(15)

\[
+ \int_{\alpha_0}^{\alpha_1} \left[ \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] d\alpha = \int_{\alpha_0}^{\alpha_1} \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} \, d\alpha.
\]

Theorem. The non-holonomic dynamical system defined by (1) and (2) admits the relative integral invariant (15).

5. A Dynamical Theorem. We have seen that the equations of motion of a non-holonomic system can be written in the Hamil-
tonian form (4). Suppose that we consider an arbitrary system of differential equations

\[
\frac{dq_s}{dt} = Q_s, \quad \frac{dp_s}{dt} = P_s,
\]

where the \(Q's\) and \(P's\) are any functions of \((q_s, p_s, t)\) with the desired continuity. Let \(a_s(q, t), a_s(q, t), \lambda_s(t)\), and \(H(q, p, t)\) be given functions, with continuity as required, and suppose that

\[
\int_{\alpha_1}^{a_1} d\alpha \int_{\alpha_0}^{a_0} \rho \left[ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial \alpha} + a_s \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+r}}{\partial \alpha} \right] \lambda_s d\alpha
\]

\[
+ \int_{\alpha_1}^{a_1} \left[ \sum_{s=1}^{n+k} p_s \frac{\partial q_s}{\partial \alpha} - H \frac{\partial t}{\partial \alpha} \right] d\alpha
\]

is a Cartan integral invariant of the system (17). This means that \(u = u_1\) determines a certain simple closed circuit \(C\) on the tube of trajectories of (17), and that the integral (18) is independent of \(u_1\). On differentiating with respect to \(u_1\), and dropping the subscript, we have

\[
\int_{\alpha_1}^{a_1} \rho \left[ \sum_{i=1}^{n} a_{ri} \frac{\partial q_i}{\partial t} + a_s \frac{\partial t}{\partial \alpha} + H \frac{\partial t}{\partial \alpha} - \frac{\partial q_{n+r}}{\partial \alpha} \right] \lambda_s d\alpha
\]

\[
+ \sum_{i=1}^{n+k} \left\{ \frac{dp_s}{dt} \frac{\partial q_s}{\partial \alpha} + p_s \frac{\partial q_s}{\partial \alpha} - \frac{dH}{dt} \frac{\partial t}{\partial \alpha} - H \frac{dH}{\partial \alpha} \right\} d\alpha = 0.
\]

Integrating by parts, we find

\[
\int_{\alpha_1}^{a_1} \rho \left[ \sum_{i=1}^{n} \left\{ \frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} + \sum_{r=1}^{k} a_{ri} \lambda_r \right\} \frac{\partial q_i}{\partial \alpha} + \frac{dH}{dt} \frac{\partial q_i}{\partial \alpha} \right] d\alpha
\]

\[
+ \sum_{r=1}^{k} \left\{ \frac{dp_{n+r}}{dt} + \frac{\partial H}{\partial q_{n+r}} - \lambda_r \right\} \frac{\partial q_{n+r}}{\partial \alpha} + \sum_{s=1}^{n+k} \left\{ - \frac{dq_s}{dt} + \frac{\partial H}{\partial p_s} \right\} \frac{\partial p_s}{\partial \alpha}
\]

\[
+ \left\{ \sum_{r=1}^{k} a_{r} \lambda_r - \frac{dH}{dt} + \frac{\partial H}{\partial t} \right\} \frac{\partial t}{\partial \alpha} \right] d\alpha = 0,
\]

or
\[ \int_C \rho \left[ \sum_{i=1}^{n} \left\{ \frac{d\dot{p}_i}{dt} + \frac{\partial H}{\partial q_i} + \sum_{r=1}^{k} a_r \lambda_r \right\} dq_i 
\]
\[ + \sum_{r=1}^{k} \left\{ \frac{d\dot{p}_{n+r}}{dt} + \frac{\partial H}{\partial q_{n+r}} - \lambda_r \right\} dq_{n+r} \]
\[ + \sum_{s=1}^{n+k} \left\{ - \frac{dq_s}{dt} + \frac{\partial H}{\partial \dot{p}_s} \right\} d\dot{p}_s \]
\[ + \left\{ \sum_{r=1}^{k} a_r \lambda_r - \frac{dH}{dt} + \frac{\partial H}{\partial t} \right\} dt \right] = 0. \]

But \( \rho \) is a positive function, and, from the way in which it was introduced (see (12b)), it is arbitrary. Hence we may infer that each parenthesis vanishes separately. The resulting equations are precisely (4) and (5), those of a non-holonomic Hamiltonian system. Accordingly we have the following result.

**Theorem.** If the integral (18) is a Cartan integral invariant of the system of differential equations (17), for a given set of \( \lambda \)'s, \( a_r \)'s, and \( a_s \)'s, and a given function \( H \), then the system (17) is of the form (4), and consequently there is a non-holonomic dynamical system for which (17) are the equations of motion.

The result is that the non-holonomic system is completely characterized by the integral invariant (18), which may be written more suggestively in the form

\[ \int_C \int_0^t \sum_{r=1}^{k} \lambda_r \left\{ \sum_{i=1}^{n} a_i dq_i + a_s dt - dq_{n+r} \right\} dt \]
\[ + \int_C \sum_{s=1}^{n+k} p_s dq_s - H dt, \]

it being understood that the upper limit \( t \) is a function of \( \alpha \) determined by the curve \( C \).

**California Institute of Technology**