

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

310. Miss M. G. Humphreys: *The representation of integers as sums of values of quartic polynomials*. Preliminary report.

The most general polynomial of degree r in x which represents integers for $x \geq 0$ is of the form $P_r(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0$. When $r=4$ and $(a_4, \dots, a_1) = 1$, $a_0 = 0$, $a_4 > 0$, any sufficiently large integer is the sum of twenty-one values of $P_4(x)$ in all cases except when the coefficients satisfy one of three sets of conditions, of which the following is an example: $3/(a_4, a_3)$, $9 + a_4$, $a_3 \equiv 6a_1 \pmod{9}$, $a_4 \equiv 6a_2 \pmod{9}$. The method used is similar to that of E. Landau in *Zum Waringschen Problem, Dritte Abhandlung* (Mathematische Zeitschrift, vol. 32 (1930), pp. 699-702). (Received August 7, 1934.)

311. Mr. J. F. Randolph: *Carathéodory measure and a generalization of the Gauss-Green lemma*.

The Gauss-Green lemma for the plane connects the double integral of a partial derivative of a function over a region R with the line integral of the function around the curve C bounding R . In the past many investigations have been concerned with the kind of regions and boundaries for which the lemma is valid. With the exception of a paper by Schauder, the boundary has been assumed to be a curve with a tangent almost everywhere. The present paper contains what seems to be extreme simplification of the conditions on the boundary. The Gauss-Green lemma is shown to hold for any simply connected region whose boundary has Carathéodory linear measure finite. Then by methods which have the effect of the usual cross cut scheme, applicable regions are extended to a wide class not simply connected. In the new auxiliary material is included the fundamental theorem that the inner Carathéodory linear measure of a set is the upper limit of the Carathéodory linear measure of closed components of the set. (Received September 26, 1934.)

312. Mr. Fritz Herzog: *Systems of algebraic mixed difference equations*.

The decomposition theory for systems of algebraic differential equations, developed by J. F. Ritt in his Colloquium Publication, and of algebraic difference equations, developed by Ritt and Doob, suggested a similar investigation

in the case of algebraic mixed difference equations. The author proves the result, analogous to that obtained in the two cases mentioned above, that a system of algebraic mixed difference equations can be decomposed in essentially one way into a finite set of irreducible systems. (Received September 27, 1934.)

313. Dr. A. B. Brown: *Functional dependence.*

Consider m functions of n variables and p parameters, with matrix of first partial derivatives of rank $< m$. Nowhere do we demand that a largest minor $\neq 0$ is not zero at a particular point. We prove, in the real case, that there is a functional relation in the large if the functions are sufficiently differentiable, generalizing a result of K. Knopp and R. Schmidt (1926), whose proof for their case $n \leq m$ admits no obvious extension to our case. If $m=2$ and the functions are analytic (in complex space) we obtain an analytic relation in the small, generalizing a result of Bliss (1913), who treated the case $m=n=2$. Our proof is different from his. For general m, n, p and analytic functions of complex variables, Osgood showed (1916) that no analytic relation need exist. We prove that a property holds which would be valid if there were an analytic relation, namely, that the values of the given functions form a set which is nowhere dense in their space and cannot disconnect any region. (Received September 24, 1934.)

314. Dr. S. E. Warschawski: *On the higher derivatives at the boundary in conformal mapping.*

Let $w=f(z)$ map the circle $|z| < 1$ conformally on the interior of a closed Jordan curve C . Conditions for the existence of $d^n f(z)/dz^n$ at the boundary, similar to those obtained in a previous paper for the case $n=1$, are obtained for every $n > 1$. Let $\Theta(s)$ be an inclination angle of the tangent line ($s = \text{arc length}$). Set $\kappa^{(n)}(s) = d^n \Theta(s)/ds^n$. If C has "L-curvature" at $w_1 = f(z_1)$, $s = s_1$, of order $n-1$, i. e., if $\lim_{s, s' \rightarrow 0} [\kappa^{(n-2)}(s) - \kappa^{(n-2)}(s')]/(s-s')$ exists, and if (*) $\int_0^a |\kappa^{(n-2)}(s+t) + \kappa^{(n-2)}(s-t) - 2\kappa^{(n-2)}(s)| dt/t^2$ converges, then $f^{(n-1)}(z)$ assumes continuous boundary values on $|z|=1$ near $z=z_1$ and is differentiable at $z=z_1$. If $\kappa^{(n-1)}(s)$ is continuous on C and (*) approaches zero uniformly on C with a , then $f^{(n)}(z)$ is continuous in $|z| \leq 1$. It is shown how "the modulus of continuity" of $f^{(n)}(z)$ on $|z|=1$ depends on $\kappa^{(n-1)}(s)$. For every $n \geq 1$, conditions are obtained under which $f^{(n)}(z)$ varies continuously in $|z| \leq 1$ under a suitable deformation of C . (Received September 29, 1934.)

315. Dr. I. J. Schoenberg: *Extensions of theorems of Descartes and Laguerre to the complex domain.*

Let $f(x) = a_0 + a_1x + \dots + a_nx^n = 0$ be an algebraic equation with real or complex coefficients. Mark the points a_ν in the complex a -plane. Draw in this plane through its origin any two lines Δ, Δ' defining two opposite angles A and B of common aperture ψ ($0 < \psi \leq \pi$) such that none of the points a_ν shall lie in the interior of either A or B . Call C and D the remaining opposite angles (of aperture $\pi - \psi$) defined by Δ and Δ' , which contain all the points a_ν in their interior or their boundary. Call $v_a(A, B)$ the number of "variations" from the

domain C to the domain D or vice-versa in the sequence of coefficients a_0, a_1, \dots, a_n , where as usual vanishing coefficients are to be disregarded. Then $v_a(A, B)$ is an upper bound for the number of zeros of $f(x)$ within the angular domain: $|\arg x| < \psi/n$. For real equations we can take $\psi = \pi$ and our result reduces to the extension of Descartes' rule given by N. Obreschkoff (Comptes Rendus, Paris, vol. 177 (1923), pp. 102-104). Rotating the x -plane by writing $x = e^{i\theta}z$, the above result gives information about the distribution of the arguments of all the zeros of $f(x)$ from the arguments of the coefficients of $f(x)$ only. Extensions to a class of exponential sums are given. Furthermore it is shown that a theorem of Laguerre (Oeuvres, vol. 1, p. 41) actually applies to the roots in certain half-planes and circles of the complex plane. Cauchy's index theorem supplies the proofs. (Received September 29, 1934.)

316. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

A vector function of the real variable t is said to be of the almost periodic form if it can be expressed as a finite sum of terms each of which is the product of an exponential function, a non-negative integer power of t , and an almost periodic vector function. This paper gives certain sets of necessary and sufficient conditions that all of the solutions of a linear system (which may or may not be homogeneous) of ordinary differential equations with almost periodic coefficients should be of the almost periodic form. The paper also gives certain sufficient conditions that a particular solution be of the almost periodic form. (Received September 27, 1934.)

317. Mr. Walter Prenowitz: *The characterization of plane collineations.*

Let R be a region of the euclidean plane. A set of line intervals contained in R is called a family of lines in R , if the ends of each interval of the set are not in R , and each point of R is on exactly one interval of the set. If four families of lines are contained in R and no two have a line in common they are said to form a 4-web in R . The principal result of this paper is as follows: Any topological transformation of region R , which carries a 4-web of lines in R into a 4-web of lines, is projective. This was proved by E. Kasner (this Bulletin, vol. 9, pp. 545-546) on the assumption that the transformation is doubly differentiable. For collineations on the projective plane, the following theorem is proved: A one-to-one point transformation on the projective plane is a collineation, if it carries three independent pencils and a line not of these pencils into three pencils and a line, respectively. This is an improvement on the result stated by the author in a preliminary report (this Bulletin, vol. 38, p. 345). The author has also derived several theorems related to this, including an analogue for a region of the euclidean plane. (Received October 1, 1934.)

318. Dr. E. J. Finan: *On the number theory of certain non-maximal domains of integrity of the total matrix algebra of order 4.*

The following four matrices $\|a_{ij}\|$, ($i, j = 1, 2$), form a basis for a non-maximal domain of integrity of the total matrix algebra of order 4: In the first,

$a_{11}=1, a_{12}=a_{21}=a_{22}=0$; in the second, $a_{12}=1, a_{11}=a_{21}=a_{22}=0$; in the third, $a_{22}=1, a_{11}=a_{12}=a_{21}=0$; in the fourth, $a_{21}=k, a_{11}=a_{12}=a_{22}=0$, where k is a rational prime. It is shown that if n is any number of the domain a necessary and sufficient condition that n be a prime is that the determinant of n be a rational prime. Ideals of the domain are represented by integral matrices of order 4. This approach is due to C. C. MacDuffee. The class number of the domain is proved to be 3. (Received September 24, 1934.)

319. Dr. James Singer: *A theorem in finite projective geometry and a number theory application.*

It is well known that a finite projective plane, $PG(2, p^n)$, exists for every prime p and positive integer n . There are p^n+1 points on a line and $p^{2n}+p^n+1$ points and lines in the plane. Furthermore, if there are $m+1$ points on a line of a finite projective plane, m is a power of a prime. We prove first that in any finite projective plane there exists a collineation which carries a point P into a point P' , P' into P'' , \dots , $P^{(q)}$ into P , where $q=p^{2n}+p^n$ and the P 's are all distinct. We then show that the points and lines of the plane can be exhibited in a regular array, that is, a matrix array of p^n+1 rows and $p^{2n}+p^n+1$ columns whose elements are symbols for points, whose columns represent the points on a line, and each of whose rows is a cyclic permutation of the first row. The existence of the regular array enables us to prove this interesting theorem: A necessary and sufficient condition for the existence of $m+1$ integers d_0, d_1, \dots, d_m such that their m^2+m differences $d_i-d_j (i \neq j)$ be congruent modulo m^2+m+1 to the integers $1, 2, \dots, m^2+m$ in some order is that m be a power of a prime. (Received September 24, 1934.)

320. Dr. J. S. Frame: *On the numerical determination of tables of characteristics of finite groups.*

Between the elements of the square array, giving by rows and columns the characteristics of matrices, one from each conjugate set in each irreducible linear representation of a finite group, there exist many numerical relations valid for all finite groups. Further relations or restrictions on this table obtain for simple groups. In this paper six of the former relations and five of the latter restrictions are shown to determine uniquely a single table corresponding to each of the group orders 60 and 168. No inner structural properties of the particular group are used, except that the group is simple and has the given order. The tables so obtained are those of known simple groups. (Received October 1, 1934.)

321. Dr. Leo Zippin: *Compact zero-dimensional abelian groups.*

A complete invariant system $I(X)$ of a compact zero-dimensional group X is given, such that X and Y are isomorphic if and only if $I(X)=I(Y)$. This equivalence-problem was solved indirectly by J. W. Alexander (*On the homology groups of abstract spaces*, Annals of Mathematics, vol. 35 (1934)) by reference to the (discrete) character groups associated with them. This proof is independent of such considerations. It is shown, further, that if $I(X)=I(Y)$

then certain subgroups of a very general nature X' and Y' (of X and Y respectively) are isomorphic, and any isomorphism between them can be extended to the original groups. (Received September 29, 1934.)

322. Professor C. N. Moore: *On criteria for Fourier constants of L -integrable functions of several variables.*

In a paper presented at the last annual meeting of the Society (see this Bulletin, abstract 40-1-41) the author has given a criterion for the Fourier cos-cos coefficients of an L -integrable function of two variables. In the present paper a more general criterion, which includes the previous one and other related criteria, is given. Furthermore the theorem is extended to functions of more than two variables and the related multiple Fourier series. The results obtained constitute a generalization to multiple series of a criterion for simple series recently given by Cesari (see Bollettino della Unione Matematica Italiana, vol. 13 (1934), p. 100). (Received September 28, 1934.)

323. Mr. Garrett Birkhoff: *Lattices of equivalence relations.*

It is shown that the possible definitions of equivalence within a class of n objects constitute a lattice E_n , dually isomorphic with the lattice of the subalgebras of a Boolean algebra of order 2^n . E_n occupies in lattice theory a position like that of the symmetric group on n letters in group theory; the parallel extends to the group of automorphisms of E_n , the classification of similar elements, transitivity, order, etc. Moreover E_n is "simple" in the sense of having no proper homeomorphic image. Various abstract methods for combining lattices are discussed, and the sublattices of E_4 are enumerated. Finally, it is shown that every lattice of subgroups is isomorphic with a lattice of equivalence relations, and conversely. (Received October 1, 1934.)

324. Professor Oystein Ore: *On the foundation of abstract algebra.*

This paper contains an attempt to unify the various branches of abstract algebra. By means of the theory of structures one can obtain a basis for such a general theory including group theory, ideal theory, moduli, etc., and the various decomposition theorems of these theories may be derived from a common source. (Received September 26, 1934.)

325. Professor Harold Hotelling: *On correlation between vectors, and a function associated with the tetrad difference.*

A generalization of the correlation coefficient is introduced which is appropriate for measuring the relationship between a set of s and a set of t variates, or between a random vector in s and one in t dimensions. This vector correlation coefficient q is invariant under internal linear transformations of the variates of either set, lies between -1 and $+1$, and for $s=t$ is actually the correlation between the corresponding s -rowed determinants of the two matrices of observations. Its distribution in samples from normal populations involves s parameters, if $s \leq t$. The most important case is that in

which $s-1$ of the parameters vanish; for this case the sampling distribution of q is determined exactly, and expressions for the moments are obtained. For $s=t=2$, the vector correlation takes the form $q=(r_{13}r_{24}-r_{14}r_{23})(1-r_{12}^2)^{-1/2} \cdot (1-r_{34}^2)^{-1/2}$. The distribution provides exact probabilities, valid even for small samples, for the tetrad difference in the numerator. Such a test of significance has previously been the object of extensive investigation by means of rough asymptotic approximations to the standard error, and is needed in mathematical psychology. (Received September 11, 1934.)

326. Dr. J. L. Doob: *Note on probability.*

The von Mises theory has as its fundamental element the idea of a collective: a sequence of observations on which two conditions are imposed. The second of these conditions states the impossibility of a system in a game of chance. Using this condition von Mises has shown that a collective, considered as a sequence of mathematical elements, cannot be obtained by analytic processes, and cannot be fully specified. Such a definition has been criticized by Kamke in a recent paper. To avoid this difficulty a modification is made in the two conditions of von Mises to make them not conditions on a fixed sequence of observations but conditions on the mechanism of the experiment. The two conditions of this modified theory correspond to theorems in the classical theory. The first condition corresponds to the strong law of large numbers. The second condition corresponds to the theorem that a system is impossible in a game of chance. A proof is given of this fact, in terms of the classical theory. By means of these considerations, an analysis is made of the relations between the von Mises theory, the modified theory, and the classical theory. (Received September 26, 1934.)

327. Mr. Garrett Birkhoff: *Linear extension of complex-valued linear functions in Banach space.*

It is shown that it is possible to define a linear complex-valued function over a plane in a three-dimensional normed vector-space, which cannot be extended without increasing the modulus. (Received September 14, 1934.)

328. Mr. Garrett Birkhoff: *On monogenic extension of linear functionals.*

It is shown that any monogenic linear functional of modulus M defined over a linear manifold of a Banach space B in which the scalars are complex numbers can be extended over B to a monogenic linear functional of modulus $\leq 2^{1/2} M$. (Received September 24, 1934.)

329. Mr. Garrett Birkhoff: *Orthogonality in normed vector spaces.*

It is proved that if in a normed vector space of three or more dimensions, the norm is differentiable, and the relation of perpendicularity is reciprocal, then the norm is Pythagorean. Nevertheless there exist infinitely many non-

isometric normed vector spaces of two dimensions which are "orthogonal."
(Received September 14, 1934.)

330. Professor H. V. Craig: *On a generalized tangent vector.*
Paper II.

By means of the vectors introduced in Paper I a metric tensor and a connection are constructed, such that the intrinsic derivative of the metric tensor is zero while the autoparallel curves are the extremals associated with F . The arguments of F are the coordinate variables and their derivatives up to order m . (Received September 28, 1934.)

331. Mr. R. M. Robinson: *The Bloch constant \mathfrak{A} for a schlicht function.*

Consider the functions $f(x)$ which are regular within the unit circle, for which $f'(0)=1$, and which map the unit circle on a plane region. Then it is known that a circle of radius $9/16$ can be found within the map. This is one of a number of results in a paper by Landau (*Mathematische Zeitschrift*, vol. 30 (1929), pp. 608-634). The upper bound of the constants which can replace $9/16$ in this statement defines the constant \mathfrak{A} , so that $\mathfrak{A} \geq 9/16$. From above the best inequality known was $\mathfrak{A} \leq \pi/4$, which was given by Szegö. By mapping the unit circle on a circle with radial slits, the author obtains the following results: using a single slit extending to the origin, $\mathfrak{A} < 3/4 < \pi/4$; using three slits extending half way to the origin, $\mathfrak{A} < 11/16$; using four or six slits suitably chosen, $\mathfrak{A} < 2/3$. (Received September 24, 1934.)

332. Professor G. T. Whyburn: *Concerning curves of finite degree and local separating points.*

In this paper a study is made of curves of finite degree in the sense of H. Kamiya (see *Tôhoku Mathematical Journal*, vol. 34 (1931), pp. 58-72) in connection with their local separating points. It is shown that a compact continuum K will be of finite degree if and only if every subcontinuum of K contains uncountably many local separating points of K . The property of being of finite degree is cyclicly extensible and reducible, and any continuum admits a *finest* upper semi-continuous decomposition so that the hyperspace is a curve of finite degree. Also it is shown that, whereas in any curve of Menger-Urysohn order ≤ 3 , *degree* and *order* are identical for all points, there exists a curve of order 4 which is of infinite degree at every point. (Received September 26, 1934.)

333. Professor G. T. Whyburn: *A decomposition theorem for continua.*

Let P be any local property of a continuum such that the set of all non- P points of any compact continuum is either vacuous or of dimension >0 . The theorem is proved that if K denotes the set of all non- P points of a compact continuum M and M is decomposed upper semi-continuously into the components of \bar{K} and the points of $M - \bar{K}$, then every point of the hyperspace H

is a P -point of H . Taking the properties of being (i) locally connected, (ii) regular, (iii) rational, (iv) of dimension $< n$, or (v) of belonging to no continuum of convergence, or (vi) of belonging to no continuum of condensation for P , we obtain that the hyperspace is (i) locally connected, (ii) a regular curve, (iii) a rational curve, (iv) of dimension $< n$, (v) hereditarily locally connected or (vi) without continua of condensation, respectively. Number (i) is the well known result of R. L. Moore concerning the decomposition of a continuum into its prime parts. (Received September 26, 1934.)

334. Professor H. S. White: *The ten common secants of two twisted cubic curves.*

These have been enumerated by Cremona on degenerate curves, by Halphen with the use of metric auxiliaries, and by Segre employing a quadric locus in five-space. The author finds a simple way to reach the enumeration, with two others of some interest, by binary correspondences in three-space and generic curves in random position, the whole process being projective. (Received October 9, 1934.)

335. Mr. Max Zorn: *On infinite algebra.*

Instead of the well-ordering principle a certain inductive principle equivalent to it is introduced and one obtains a considerable simplification in the proofs of Steinitz' theorems on transfinite algebra. (Received October 3, 1934.)

336. Mr. W. E. Sewell: *Generalized derivatives and approximation by polynomials.*

Let $f(z)$ be analytic in the simply connected region R bounded by the analytic Jordan curve C and continuous in the corresponding closed region; then Riemann's definition of a generalized derivative on the axis of reals is extended to $f(z)$ on C . The existence of a generalized derivative under conformal transformation is preserved. Bernstein's theorem on the modulus of the derivative of a polynomial is generalized. Sufficient conditions that $f(z)$ have a generalized derivative are established in terms of the degree of approximation of polynomials and also in this connection more general point sets are considered. The relation between Hölder conditions and generalized derivatives is considered in connection with approximation. An example of the results established is the following: Theorem. *If for every n there exists a polynomial $P_n(z)$ of degree n , such that $|f(z) - P_n(z)| < M/n^\alpha$, $\alpha > 0$, on and within an analytic Jordan curve C , and $f(z)$ is analytic interior to C , then $f(z)$ has derivatives of all orders α' less than α on the curve C .* (Received September 11, 1934.)

337. Mr. Garrett Birkhoff: *Operational identities on point sets.*

There exist operational identities which perfectly characterize systems of point sets closed relative to (i) finite sums and products, (ii) complements and finite sums and products. This note shows that no such operational identities

can exist relative to (iii) enumerable sums and products, (iv) transfinite sums and products, (v) complements and enumerable sums and products. (Received October 3, 1934.)

338. Dr. Solomon Kullback: *On the Bernoulli distribution.*

By employing merely the law of combination of probabilities we are enabled to derive in an elementary manner, without the use of characteristic functions or the explicit form of the distribution itself, the expected values of the moments of the Bernoulli distribution. The discussion is readily extended to the Poisson exponential distribution and to distributions of the Lexis and Poisson type. (Received October 11, 1934.)

339. Mr. Garrett Birkhoff: *Ideals in algebraic rings.*

The main result of the paper is as follows: Let x be any algebraic integer, $f(x)=0$ the irreducible equation for x , and $R(x)$ the ring generated by x and the rational integers. Then the conditions that the ideals of $R(x)$ —except zero—should (1) satisfy the distributive law relative to g.c.f. and l.c.m., (2) be “regular” in the sense of Grobner, and (3) be of the canonical type described by E. Noether, are the same, and are (4) that if $g(x)$ is any irreducible factor of $f(x)$ modulo any prime p , then either $g^2(x)$ is not a divisor of $f(x)$ modulo p , or else the g.c.f. of the ideals $g(x)R$ and p^2R divides pR . (Received October 17, 1934.)

340. Professor R. S. Burington: *On the equivalence of quadrics in m -affine n -space and its relation to the equivalence of $2m$ -pole networks.*

The work of Caer (Göttinger Nachrichten, 1934) and others on the equivalence of $2m$ -pole electrical networks indicates the importance of the matrix study of quadratic forms F under m -affine non-singular transformations, $x_i = x'_i$, $i=1, \dots, m$, $x_j = \sum_{k=1}^n b_{jk} \cdot x'_k$, $j=m+1, \dots, n$, of matrix T . Two symmetric matrices A and B , of forms F and G , are m -affine congruent if and only if there exists a T such that $A = T' \cdot B \cdot T$. It is assumed that the elements of the above matrices belong to a field. A set of integer, absolute, and matrix invariants is exhibited in terms of which necessary and sufficient conditions for the m -affine congruence of A and B are given. A detailed reduction for the 4-pole case is given; $m=0$ yields the projective theory, $m=1$, the usual affine case. (See R. S. Burington, American Mathematical Monthly, vol. 39 (1932), pp. 522–32; Physical Review, vol. 45, p. 429.) The relation of this work with that of Caer is discussed. (Received October 31, 1934.)

341. Professor S. S. Cairns: *Triangulation of the manifold of class 1.*

A construction is given for triangulating the manifold of class 1 as defined by Veblen and Whitehead in *The Foundations of Differential Geometry* (Cambridge Tract No. 29, 1932, Chapter VI). The method employed is a refinement of that developed in the writer's earlier paper on the triangulation of regular

loci (Annals of Mathematics, vol. 35 (1934), pp. 579–587). (Received November 7, 1934.)

342. Dr. L. A. Dye: *Space involutorial transformations of the Geiser and Bertini types.*

One form of generalization of a plane involution is a space involutorial transformation in which each plane of a pencil is invariant and in each such plane there is a plane involution of the same type. Transformations of the Geiser and Bertini types are discussed in this paper by means of a mapping on a cubic surface. The signature of the Bertini transformation is $I_{120n+61}$; $J_{120n+34+6t} + (O, \bar{O})_{120n+40} + C_{12n+6}^6$; the Geiser transformation has the signature I_{24n+19} ; $I_{24n+11+3t} + O_{24n+14} + C_{12n+6}^3$. The fundamental points O, \bar{O} lie on the fundamental line l which meets the curve C_{12n+6} $12n$ times. (Received November 7, 1934.)

343. Professor Orrin Frink: *Differentiation of sequences.*

Conditions under which it is permissible to differentiate term by term a convergent sequence of functions can be found in the literature, but they usually include the hypothesis that the derived sequence converges. Without this hypothesis, a number of conditions, one of them being the equicontinuity of the derived sequence, are here shown to be sufficient for termwise differentiability. Some of the conditions found are, like that of equicontinuity, connected with the question of the compactness of sets of functions for different types of convergence. (Received November 3, 1934.)

344. Dr. M. C. Hartley: *On the mapping of the 24-tuples of the involutorial G_{24} in a plane upon a quadric.*

The permutations of the quantities $\pm x_1, \pm x_2, \pm x_3$ considered as projective coordinates in a plane determine 24 points in the plane and hence an involution of 24-tuples which may be mapped on a rational surface. If ϕ_1, ϕ_2, ϕ_3 are the elementary symmetric functions of x_i^2 , $y_i = a_i\phi_1^4 + b_i\phi_1^2\phi_2 + c_i\phi_1\phi_3 + d_i\phi_2^2$ is a symmetric function of degree 8. There are then 4 linearly independent functions y_i ($i=1, 2, 3, 4$) which may be set proportional to the 4 projective coordinates of a point in space. To every 24-tuple corresponds, in a (1, 1) manner, a point (y). The locus of these points (y) is proved to be a quadric cone, whose equation is found and discussed. The mapping of this cone upon the plane leads to properties of plane curves of order $8n$, and to theorems on octics. (Received November 7, 1934.)

345. Dr. M. R. Hestenes (National Research Fellow): *A basic theorem in the problems of Lagrange and Bolza.*

In the present paper we prove the following theorem: Let E_{12} be an extremal arc in the problem of Lagrange or Bolza in the calculus of variations. If E_{12} satisfies the strengthened Clebsch condition and has on it no point conjugate to its initial point 1, then E_{12} is an extremal of a Mayer field. With the help of this result one can obtain the classical sufficiency theorems for these problems

without assumptions regarding normality. In this sense the present paper completes the results recently obtained by the author for the general problem of Bolza (Transactions of this Society, October, 1934). The theorem here given has been established recently by Morse (Abstract 40-9-303) and by Reid (Abstract 40-9-306). The method here used is different from those of Morse and Reid and is of interest since we show that the theorem here given is an easy consequence of certain results established by the author for the general problem of Bolza. (Received October 24, 1934.)

346. Dr. Rosella Kanarik: *Fundamental regions in S_4 for the Hessian group.*

In this paper the fundamental regions for the group of 216 in three variables are determined. Nine Hermitian forms which form a single set of conjugates under the group are chosen. Points which do not lie on the 36 hypersurfaces obtained from these forms are considered. Instead of the expected $9!$ arrangements of values according to order of magnitude of the nine definite forms, only 19,008 possible arrangements can be found. The substitutions $x_2/x_1 = re^{i\theta}$ and $x_3/x_1 = se^{i\phi}$ are made in the forms. That the first form may be selected in any one of nine ways and the second in any one of eight ways is shown by the substitution of coordinates of points in the forms. The third form, however, can be selected in only six ways. The remaining forms give rise to only 44 arrangements. Thus, since every arrangement has 216 conjugates under the group, there are 216 fundamental regions composed of 88 sub-regions. (Received November 5, 1934.)

347. Professor N. H. McCoy: *On the rational canonical form of a function of a matrix.*

Let A be a given matrix of order n with elements in the field F of complex numbers, and $\phi(\lambda)$ a polynomial with coefficients in F . The problem of determining the elementary divisors of $\phi(A)$ from those of A has been discussed by Kreis, Krishnamurthy, Turnbull and Aitken, Rutherford and Amante. The purpose of the present paper is to solve the corresponding problem for the case in which the field F is replaced by an arbitrary field K . A brief application is given to the rational solution of scalar matrix equations. (Received October 19, 1934.)

348. Professor E. G. Olds: *Distributions of greatest variates, least variates, and intervals of variation in samples from a rectangular universe.*

Neyman and Pearson have given the distribution of intervals of variation for samples from a continuous, rectangular universe. Rider has considered an infinite, rectangular population of discrete variates, and obtained the distribution of intervals of variation for samples of four from a ten-class universe. The present paper gives the distribution of intervals of variation for samples drawn, without replacement, from the population characterized by the frequency distribution $f(x) = 1$ for $x = 0, 1, 2, \dots, b$, (and zero elsewhere), and

compares the distribution with those mentioned above. It also derives the distributions of greatest and of least variates, for samples from this same universe, and compares them with those obtained for a continuous universe. (Received November 1, 1934.)

349. Professor W. E. Roth: *On k -commutative matrices.*

(a) The matrix B_k , defined by the recurrence formula $B_{i+1} = AB_i - B_iA$, ($i=0, 1, \dots, k-1$), where A and $B = B_0$ are given $n \times n$ matrices, is the k -commute of A with respect to B . (b) The matrix, A , is k -commutative with respect to B if $B_k = 0$ and $B_{k-1} \neq 0$. According to these definitions A and B are commutative in the usual sense if $B_1 = 0$; and quasi-commutative in the McCoy sense (*On quasi-commutative matrices*, Transactions of this Society, vol. 36 (1934), pp. 327-340) if A and B are mutually 2-commutative. Mutually k -commutative matrices, $k > 2$, likewise exist. The present paper exposes certain properties of k -commutative matrices and of mutually k -commutative matrices and develops the algebra of such matrices. (Received November 1, 1934.)

350. Professor J. L. Synge: *Principal null-directions in space-time defined by an electromagnetic field.*

In a flat space with fundamental form $dx_r dx_r$, any symmetric tensor T_{rs} defines principal directions ξ_r and invariants X by means of the equations $T_{rs} \xi_r + X \xi_s = 0$. The paper investigates these directions when T_{rs} is the energy tensor of an electromagnetic field. It is found that the four invariants are of the form $-k, +k, -k, +k$, where k differs from zero if and only if the field is not null, a null-field being one in which the electric and magnetic vectors are equal and perpendicular. The principal directions are partially indeterminate, but among them there exist two principal null-directions in general, degenerating to one when the field is null, and in that case represented to any observer by a point travelling along the pointing vector with the velocity of light. In the general case the two principal null-directions define a 2-flat, and if the time-axis is taken in it, the electric and magnetic vectors have a common line of action. (Received November 7, 1934.)

351. Professor J. L. Synge: *Some intrinsic and derived vectors in a Kawaguchi space.*

In a Kawaguchi space of order m the length of a curve $x^i = x^i(t)$ is $\int_{t_1}^{t_2} F dt$, where F is a function of the parameter t and of the coordinates and their derivatives with respect to t up to the m th order. It is shown that any curve defines a set of $m+1$ vectors $E_i^p = \sum_{q=p}^m (-1)^q C_p (d^{q-p}/dt^{q-p}) (\partial F / \partial x^{(q)i})$, ($p=0, 1, \dots, m$), where $x^{(q)i} = d^q x^i / dt^q$; some of these have already been found by H. V. Craig. If $m=1$ (Finsler space), the two vectors of this type are well known: $E_i^0 = F_{(0)i} - dF_{(1)i}/dt$, $E_i^1 = -F_{(1)i}$. Other intrinsic vectors are also obtained. For an assigned vector field X^i , the invariant $X^j E_j^p$ is used as a "generating function" to obtain a set of covariant vectors involving X^j and its derivatives, $D_{i^r}^{p,j} X^i = \sum_{p=r}^{2m-p} (-1)^q C_r (d^{q-r}/dt^{q-r}) (x^j E_j^p)_{(q)i}$, ($p=0, 1, \dots, m; r=0, 1, \dots, 2m-p$). One of these formulas (slightly modified)

enables us to define parallel propagation in a Kawaguchi space. (Received November 7, 1934.)

352. Professor W. J. Trjitzinsky: *Laplace integrals and factorial series in the theory of linear differential and linear difference equations.*

In the fields of ordinary linear differential and linear difference equations, whose coefficients are convergent series in negative powers of $x^{1/p}$ (integer $p \geq 1$) with, possibly, a few positive powers present, a fundamental problem is that concerning the nature of the solutions in the vicinity of the singular point ($x = \infty$). The method of Laplace integrals, leading to convergent factorial series developments (summation of formal series solutions), is not applicable to the unrestricted problem, the latter being treated completely by asymptotic methods in recent papers by G. D. Birkhoff and by the author (*Acta Mathematica*). However, the method of Laplace integrals, whenever applicable, is more satisfactory than the asymptotic method. In the present work application of the stated method is carried considerably beyond the results of earlier writers (notably, Nörlund and Horn). "Gegenbeispiele" are given making it evident that, in so far as the essential features of the theory are concerned, the main theorems established are substantially of the greatest possible generality. This work will appear in the *Transactions of this Society*. (Received November 7, 1934.)

353. Professor G. T. Whyburn: *Concerning K -perfect sets.*

Let K be any class of closed sets. A set A will be said to be K -perfect provided A is closed and $K(A) = A$, where $K(A)$ denotes the K -derivative of A as defined in my paper in the *American Journal of Mathematics*, vol. 54 (1932), p. 170. In this paper it is shown that if C is any separable and metric space, then for any class K , C is the sum of a K -perfect set and a countable number of K -sets. Also relations are established between the classes of all K -perfect sets corresponding to various choices of the class K . (Received November 9, 1934.)

354. Professor R. L. Wilder: *A characterization of manifold boundaries in E_n dependent only on lower dimensional connectivities of the complement.*

This paper is a sequel to our recent paper *Generalized closed manifolds in n -space* (*Annals of Mathematics*, October, 1934), to which we refer for definitions of terms used. In E_n ($n \geq 3$), let a compact point set M be a common boundary of (at least) two domains D_1 and D_2 such that D_k ($k = 1, 2$) is uniformly locally i -connected for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 3$. Then, if one of the numbers $p^{n_k+1}(D_k)$ is finite, M is a generalized closed $(n-1)$ -manifold. This contains Principal Theorem E of our former paper as a special case (as well as Theorem 20, *Mathematische Annalen*, vol. 109, p. 305), showing, incidentally, that the restriction on $p^{j+2}(D)$, \dots , $p^{n-2}(D)$ in that theorem is unnecessary. The condition that a $p^{n_k+1}(D_k)$ be finite may be replaced by the

condition that for one D_k there is an $\epsilon > 0$ such that $(n_k + 1)$ -cycles of D_k of diameter $< \epsilon$ bound in D_k . We also obtain extensive generalizations of Theorems 11 and 15 of our *Mathematische Annalen* paper referred to above. (Received November 9, 1934.)

355. Professor H. J. Ettliger: *Linear derivative inequalities.*

By the method of the integrating factor, the first order linear inequality, $v' + P(x)v \leq Q(x)$, where $v(x)$ is an absolutely continuous function on $(0, 1)$, and $P(x), Q(x)$ are integrable (Lebesgue) on $(0, 1)$, is solved. The result for $Q(x) \equiv 0, v(0) = 0$ leads immediately to establishing the uniqueness of an absolutely continuous solution for the general first order system satisfying a Lipschitz condition. The result for the non-homogeneous relation establishes at once the continuity of the solution of the general first order system as a function of the coefficients of the system and the set of initial values. (Received October 23, 1934.)

356. Mr. F. S. Harper: *On a function to represent infantile mortality.*

Many efforts have been made to increase the range of ages for which Makeham's function will satisfactorily represent the $\log l_x$, where l_x represents the number living at age x , because of certain theoretical and practical advantages principally in the computation of joint life functions. Difficulties are met at both extremes of the mortality table; however, the infant ages present the greater problem due to the very rapid decrease in l_x during the first year of life. In the present paper a function is introduced which represents $\log l_x$ by months in the first year of life with sufficient accuracy to reproduce the recorded values of l_x for one of the U. S. Life Tables with an error of less than 1 in 100,000. This function is then used in conjunction with Makeham's to reproduce l_x from age 0 to 81 with a maximum error of two per cent. An expression valid for the whole range of life has been found by introducing an auxiliary three parameter function to represent the error from age 81 to the end of the table. Formulas are developed to be used in the computation of joint life functions involving two lives. (Received October 24, 1934.)

357. Dr. Solomon Kullback: *On the multinomial distribution.*

The notions introduced in a previous paper, *On the Bernoulli distribution*, are applied to derive the expected values of the moments and product moments of the multinomial distribution. The results are very simply formulated in the symbolism of the calculus of finite differences. (Received October 26, 1934.)

358. Mr. G. D. Nichols: *The arithmetized expansions for certain doubly periodic functions of the third kind.*

The explicit arithmetized Fourier series developments are obtained for $\theta_1'^3 \theta_\alpha(x+y) / \theta_\beta^3(x) \theta_\gamma(y)$ and $\theta_1'^3 \theta_\alpha^2(x+y) / \theta_\beta^3(x) \theta_\gamma^2(y)$, where the θ 's are the Jacobi theta functions, x and y are independent complex variables, and (α, β, γ) are a certain sixteen triads out of a possible sixty-four which can be

selected from 0, 1, 2, 3. The arithmetical results obtainable from these expansions involve incomplete numerical functions in two variables. Some of these results are given. (Received October 29, 1934.)

359. Mr. K. S. Ghent: *A generalization of Waring's theorem.* Preliminary report.

R. E. Huston (in his doctor's thesis, University of Chicago) has shown that the s used in the first Hardy-Littlewood theory suffices for the more general function $f = a_1 x_1^k + \dots + a_s x_s^k$ if certain congruential conditions are satisfied by the coefficients a_1, \dots, a_s . It can now be shown that the smaller s of the second Hardy-Littlewood theory will suffice for the same function under the same congruential conditions. (Received October 16, 1934.)

360. Dr. M. S. Robertson (National Research Fellow): *Stieltjes integral for functions convex in one direction.*

In this paper are considered the class \mathfrak{F} of analytic functions $f(z) = z + \sum_2^{\infty} a_n z^n$, a_n real, univalent, and *convex in the direction of the imaginary axis* for $|z| < 1$, i.e., every straight line parallel to the imaginary axis cuts the image of the circle $|z| = r$, for every r in the interval $(0, 1)$, in not more than two points. The author shows that the necessary and sufficient condition that $f(z)$ belong to class \mathfrak{F} is that it can be represented in the form $f(z) = (1/4\pi i) \int_{-\pi}^{\pi} [\log(1 + ze^{i\theta}/1 + ze^{-i\theta})] d\alpha(\theta)/\sin \theta$ where $\alpha(\theta)$ is an odd, non-decreasing function of θ in $(-\pi, \pi)$ given by the series $\alpha(\theta) = \theta + \sum_1^{\infty} [(k+1)a_{k+1} - (k-1)a_{k-1}/k] \sin k\theta$. Various properties of $f(z)$ are deduced from this representation. (Received October 26, 1934.)

361. Professor A. T. Craig: *On the distribution of a certain product moment.*

In the present paper, the distribution function of the product moment $u = \sum x_j y_j / N\sigma_x \sigma_y$ is found when x_j and y_j ($j=1, 2, \dots, N$) are $N=2k$ values of two normally correlated variables x and y . If the coefficient of correlation between x and y is zero, the distribution function of u is identical with that of the arithmetic mean of samples of k independent items drawn from a population characterized by Laplace's first law of probability. (Cf. Mayr, Monatshefte für Mathematik und Physik (1920), p. 25). (Received November 1, 1934.)

362. Mr. A. P. Cowgill: *On the summability of a class of series of Jacobi polynomials.*

This paper proves that the series $\sum_{n=1}^{\infty} n^i [(p+1)(p+3) \dots (p+2n-1)/2^n n!] \cdot X_n(p, (p+1)/2, (1-x)/2)$, $-1 < p \leq 1$, where X_n is a Jacobi polynomial and i is a positive integer, is summable (C, k) , $k > i - \frac{1}{2}$, $-1 < x < 1$. Legendre polynomials, $p=1$, had previously been proved summable (C, k) , $k > i - \frac{1}{2}$. In the proof the sum of n terms of the given series is transformed by the recursion formula for Jacobi polynomials into a new sum of n terms, plus four additional terms. Convergence factors for summability $(C, i-1)$ are applied. This

causes the highest ordered part of the sum of the two additional terms involving n to take the form of a product of two series by use of the Christoffel-Darboux formula. This product is summable (C, j) , $j > \frac{1}{2}$, to the value zero, so the total order of summability is $k > i - \frac{1}{2}$. (Received November 5, 1934.)

363. Professor A. A. Albert: *Involutorial simple algebras and real Riemann matrices.*

A simple algebra A over F is said to be J -involutorial if there is a self correspondence J of A such that $(a+b)^J = a^J + b^J$, $(ab)^J = b^J a^J$, $(a^J)^J = a$, $\lambda^J = \lambda$ for every a and b of A and λ of F . The structure problem for J -involutorial simple algebras is closely connected with the study of the multiplication algebras of real Riemann matrices. The author has completed a study of both of these problems and also has shown how closely related they are to Weyl's recent generalization of Riemann matrices. A complete reduction of any impure real Riemann matrix to pure components has also been obtained. (Received November 2, 1934.)

364. Professor C. C. Camp: *A generalization of Chevalliet's formula.*

When the third derivative has the same value at both ends of the interval the error in Simpson's Rule is no longer given by the formula of M. Chevalliet. Consequently when n , the number of strips, is doubled, the second error is not approximately the difference of the computed results divided by $2^4 - 1$ as given by Scarborough (*Numerical Mathematical Analysis*, p. 161). For his case of $f(x) = 1/(1+x^2)$ the error varies inversely as n^6 . The new error accordingly is nearly the difference divided by $2^6 - 1$. By starting with $n = 10$, doubling twice and carrying the computation to 16 decimals, this ratio may be determined empirically to be near a power of 2. Then the last error may be estimated correctly to sixteen decimal places. The ratio rule may be generalized for other calculations in which n is doubled; e.g., to calculate $\sum_1^\infty u_n$ by the approximate formula $\sum_1^{n-1} u_n + \int_n^\infty u_n dn + \frac{1}{2}u_n$ for two values of n . For $u_n = n^{-p}$, $p > 1$, the error varies as n^{-p-1} . The ratio rule will apply for any convergent sequence of calculated values where the error varies asymptotically as a power of n and n is sufficiently large. (Received November 2, 1934.)

365. Professor C. C. Camp: *Note on numerical evaluation of double series.*

The Euler-Maclaurin summation formula has been extended, by Dr. Sheppard in 1900 and Irwin in 1923, to two variables to determine cubature formulas. A more complicated two-dimensional form, for which a remainder term was also calculated, was given by Baten in 1932 in the American Journal of Mathematics. The purpose of this paper is to apply the simpler formula to the numerical evaluation of double series of positive terms by the use of integrals and derivatives. If the double series converges one may sum by rows, or columns, using the ordinary sum formula twice. Such series as $\sum \sum (m^2 + a^2 n^2)^{-2}$ may be summed in this way by taking out a square block of

n^2 terms and applying the formula to the rest; $u(m, n) = (m+n)^{-p}$, $p > 2$, may be summed diagonally and thus converted to the single series $\sum n(n+1)^{-p}$. For certain double series lower and upper bounds may be interpreted geometrically in terms of volumes of truncated prisms associated with a surface $z = u(x, y)$. (Received November 2, 1934.)

366. Professor M. G. Gaba: *Finite geometries*. Preliminary report.

The usual approach to non-euclidean geometries has been through the concept of distance, and consequently non-euclidean finite geometries have been ignored. In this paper finite non-euclidean geometries are set up and it is shown that many of the theorems of the classical non-euclidean geometries hold. There are also exhibited finite geometries to fit various coordinate systems which possess peculiarities in their ideal regions. (Received November 2, 1934.)

367. Professor M. A. Basoco: *On certain systems of polynomials*.

Nörlund (Acta Mathematica, 43 (1922), pp. 121–196) has introduced certain systems of polynomials which generalize the classical Bernoulli and Euler polynomials. More recently, Milne-Thomson (Proceedings of the London Mathematical Society, 35 (1933)), has studied these in connection with generalizations of the Hermite polynomials. In the present paper, further properties of these systems are obtained through the application of theorems indicated by Appell in his discussion of a very general class of polynomials (Annales Scientifiques de l'École Normale Supérieure, 2, vol. 9, (1880)). In particular, the Fourier series representation in the interval $0 \leq x \leq 1$ of this general class of polynomials is obtained and the results are applied to the special instances due to Nörlund and Milne-Thomson. (Received November 2, 1934.)

368. Professor M. A. Basoco: *On a certain identity due to Hermite*.

In a letter to Stieltjes (*Correspondance de Hermite et Stieltjes*, vol. 2, p. 273; Lettre 263) Hermite states without proof an identity involving a certain functional of two arbitrary functions. In the present paper, similar identities involving an arbitrary number of functions are established. These find an immediate application in the theory of the doubly periodic functions of the third kind, the Jacobi theta functions, etc. From these it is possible to obtain a series of identities from which various arithmetical theorems may be deduced. (Received November 2, 1934.)

369. Professor M. A. Basoco: *Arithmetized expansions for certain pseudoperiodic functions*.

In the present paper we are concerned with the derivation of the Fourier expansions for the functions $Z_{\alpha\beta}(x, y) = \theta_1'(0, q)\theta_\alpha(2x+2y, q^2)/\theta_\beta(x, q)$, $T_{\alpha\beta}(x, y; n) = n\theta_1'(0, q^n)\theta_\alpha(x+y, q)/\theta_\beta(nx, q^n)$, ($\alpha, \beta = 0, 1, 2, 3$), where n is a

positive integer > 1 , and $\theta_\alpha(z, q)$ are the Jacobi elliptic theta functions. As is the case with developments of this sort, they are of interest in connection with arithmetical applications. It is found that the functions $Z_{\alpha\beta}(x, y)$ have an arithmetical structure related to an *indefinite* quadratic form. So far as is known no other expansions of functions of *two* variables have been given with an arithmetical structure of this type. The functions $T_{\alpha\beta}(x, y; n)$ are of interest because they involve irrationalities related to the algebraic number field $K(\sin \pi/n)$, and therefore offer the possibility of deriving (for particular values of n) arithmetical results by equating coefficients of linearly independent irrationals, in identities involving the $T_{\alpha\beta}(x, y; n)$. (Received November 2, 1934.)

370. Professor M. A. Basoco and E. T. Bell: *Further theta expansions useful in arithmetic.*

In a paper with similar title (E. T. Bell, *Messenger of Mathematics*, vol. 54 (1924), pp. 166–176) several such expansions for functions of one variable have been given, all being deduced from those of the so-called doubly periodic functions of the second kind $\theta'_1(\theta_a(x+y)/\theta_b(x)\theta_c(y))$, ($a, b, c = 0, 1, 2, 3$), where the triple index (abc) has certain sixteen values. These were first obtained by Hermite and Kronecker. For the full and efficient use of the method of paraphrase, it is necessary to have the trigonometric expansions of the remaining forty-eight functions. Following a method suggested elsewhere (M. A. Basoco, *American Journal of Mathematics*, vol. 54 (1932), pp. 242–252) these expansions in two distinct arithmetical forms have been obtained. In neither set of forty-eight expansions has it been possible to refer to the divisors of a single integer, as is the case for Hermite's sixteen. Instead, the solution in positive integers of equations of the form $n = xy + zw$, $n = xy + 2zw$, etc., are required. All sixty-four expansions, when used in paraphrase, have the desirable effect of introducing cross product terms into the quadratic forms appearing in the partitions. (Received November 2, 1934.)

371. Dr. E. W. Anderson: *Statics of special types of homogeneous elastic slabs with variable thickness.*

The derivation and solution of the fourth order partial differential equation for the deflection $z(x, y)$ of the middle surface of a thin rectangular shell of variable thickness ($t(y) = t_1 e^{cy}$, $c = (1/L) \log(t_2/t_1)$) for which the upper and lower surfaces are defined by $z = \pm t/2$ are carried out for the following cases: (1) a uniformly loaded, semi-infinite slab ($-\infty \leq x \leq \infty$, $0 \leq y \leq L$) with simply supported or clamped edges and an initially cylindrical middle surface with large radius of curvature; (2) a simply supported, rectangular plate ($0 \leq x \leq a$, $0 \leq y \leq L$) with plane central surface subjected to rectangular, line, or point loads. The solution is effected first by a complex contour integral method and then by a Fourier series method. By setting $t = \text{constant}$ the results are shown to be in agreement with known solutions of plates of uniform thickness. (Received November 5, 1934.)

372. Mr. J. Biggerstaff: *Solutions of certain ordinary and partial differential equations by means of operators.*

In volume 12 of the "Proceedings of the London Mathematical Society," Glaisher solves certain partial differential equations by using Poisson's integral $e^{a^2} = \int_{-\infty}^{\infty} e^{-u^2+2au} du$. This scheme of using a definite integral to obtain the result of operating on an arbitrary function with an operator which occurs transcendently is only one among many others which can be used. The writer has had considerable success in solving partial differential equations and systems of partial differential equations which reduce to the result of operating with a transcendental operator. The results are very easily obtained (requiring only a table of definite integrals), and are in the form of definite integrals. They are not necessarily unique. As special cases we obtain the solutions of the differential equations which occur in operational circuit analysis and which have been solved by a much longer process, usually for special cases, in the books on this subject by Bush, Berg, and Cohen. (Received October 17, 1934.)

373. Professor A. F. Moursund: *On the sum of the r th derived series of the conjugate Fourier series.*

When $f(x)$ is Lebesgue integrable on $(-\pi, \pi)$ and of period 2π , there are two equivalent Cauchy integrals either of which may serve as a formula for the sum of the conjugate Fourier series (see G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 211-246). In this note we give a direct proof of the equivalence of two analogous Cauchy integrals which we use as formulas for the sum of the r th derived series of the conjugate Fourier series. We give also a theorem concerning the Bosanquet-Linfoot summability of the r th derived conjugate series. (Received October 29, 1934.)

374. Professor E. T. Bell: *Arithmetical theorems on Lucas functions and Tchebycheff polynomials.*

A simple isomorphism between the Lucas functions, or the Tchebycheff polynomials, and the circular functions, distinct from that of Lucas (American Journal of Mathematics, vol. 1, 1878, p. 189, equation (5)) is first developed. By this means the translation of trigonometric identities into unique correspondents in Lucas functions is immediate. It is then shown that any identity in elliptic, or elliptic theta functions, implies and is implied by an identity in Tchebycheff polynomials (one complex argument) or Lucas functions (two complex arguments). (Received November 1, 1934.)

375. Dr. R. D. James: *Waring's problem for rational numbers.* Preliminary report.

Let $v(k)$ denote the least value of r such that all rational numbers are sums of r rational k -th powers ≥ 0 . It is known that $v(2) = 4$, $v(3) = 3$ (see H. W. Richmond, Proceedings of the London Mathematical Society (2), vol. 21 (1922-23), pp. 401-409), but for $k \geq 4$ nothing seems to be known. In this paper an upper bound for $v(k)$ is obtained. It was pointed out to the author by H. S. Zuckerman that any result concerning the representation of all *sufficiently large* integers as a sum of integral k -th powers implies a similar result for the representation of

all rational numbers as a sum of rational k -th powers. Hence $v(k) \leq G(k)$, where $G(k)$ denotes the least value of n such that all sufficiently large integers are sums of r integral k -th powers. Using the Hardy-Littlewood result for $G(k)$ (Mathematische Zeitschrift, vol. 23 (1925), pp. 1-37) this gives $v(k) \leq (k-2)2^{k-2} + k + 5 + \zeta_k$, where $\zeta_k = [\{(k-2) \log 2 - \log k + \log(k-2)\} / \{\log k - \log(k-1)\}]$. When k is odd and ≥ 5 the author's result for $G(k)$ (Proceedings of the London Mathematical Society (2), vol. 37 (1934), pp. 359-291) gives $v(k) \leq (k-3)2^{k-2} + k + 9 + \zeta_k$. A modification of the Gelbeke method (Mathematische Annalen, vol. 105 (1931), pp. 637-652) leads to the inequality $v(k) \leq (k-2)2^{k-2} + k + 4 + \zeta_k$. This gives, for example, $v(4) \leq 18$. (Received November 2, 1934.)

376. Dr. C. B. Morrey, Jr.: *On the differentiability of the solutions of a class of regular two dimensional minimum problems.*

Let $f(p, q)$ be a function defined, of class C'' , and satisfying $f_{pp}f_{qq} - f_{pq}^2 > 0$ for all real values of p and q , and whose second derivatives satisfy a Hölder condition; let C be a convex curve bounding the region R and let Γ be a curve above C such that any plane through three points of Γ makes an angle $\leq \tan^{-1} \Delta$ with the (x, y) plane. Radó and Haar have shown the existence and uniqueness of a surface $z = z(x, y)$ bounded by Γ and such that $z(x, y)$ satisfies a Lipschitz condition with the constant Δ and minimizes $\iint_R f(p, q) dx dy$, among all functions satisfying any Lipschitz condition on R and taking on these boundary values. The present paper proves that this solution possesses continuous first partial derivatives which satisfy a Hölder condition on every closed subregion of R . From results due to Lichtenstein and E. Hopf, we may conclude that z_{xx} , z_{xy} , and z_{yy} are continuous and satisfy a Hölder condition, and if $f(p, q)$ is analytic, we may further conclude that $z(x, y)$ is analytic. The solution $z(x, y)$ satisfies the Euler-Lagrange equation $f_{pp}z_{xx} + 2f_{pq}z_{xy} + f_{qq}z_{yy} = 0$ everywhere in R . (Received November 2, 1934.)

377. Dr. Raymond Garver: *Postulates for groups and for commutative groups.*

This paper presents a three-postulate definition of group, and a two-postulate definition of commutative group. (Received November 3, 1934.)

378. Professor E. R. Hedrick: *Generalizations of the Mittag-Leffler theorem and allied theorems.*

The usual theorem regarding the representation of an analytic function that has a finite number of singularities by means of the sum of the Laurent developments is extended in this paper to functions that are not assumed to be analytic, but only to possess continuous second partial derivatives, at points other than the singular points. A similar theorem for non-analytic functions is stated when the singularities are of a much more general type, the only restrictions being that the function behave, near each singular point, like a function that has a singularity at that point only. Finally, the Mittag-Leffler theorem that there exists a function with any preassigned Laurent developments at any preassigned set of singular points that have no finite limiting point is generalized

by replacing the Laurent development at any one singular point by a development whose real part is the real part of some Laurent development, and whose imaginary part is the imaginary part of some (other) Laurent development. (Received November 3, 1934.)

379. Mr. Ivar Highberg: *The existence of rings in vectorial spaces.*

In every vector space real or complex, not necessarily normed, there is shown to exist a symmetric bilinear function, $B(x, y)$, on E^2 to E , thus analogous to a multiplication of elements, and with the associative property $B[x, B(y, z)] = B[B(x, y), z]$. Further properties of this function are considered. (Received November 3, 1934.)

380. Mr. Ivar Highberg: *The existence of a Hermitian inner product in complex vectorial spaces.*

It is proved that in every vector space, closed under multiplication by complex numbers, but not necessarily normed, there exists a function $Q(x, y)$ on E^2 to C , which has the five properties that characterize the Hermitian inner product in Hilbert space. The relation between convergence defined by the norm $[Q(x, x)]^{1/2}$ and an arbitrarily defined norm is considered. (Received November 3, 1934.)

381. Mr. William A. Mersman: *A generalization of the Lebesgue integral.*

The purpose of this paper is to give a definition of an integral of abstract-valued functions of an abstract variable which will be a generalization of similar existing theories. This is attained by letting the range of the variable be a space in which a measure function is assumed to exist, while the values of the function are elements of a vector space. After defining a measurable function as the limit of a sequence of step-functions, we demonstrate the existence of the integral of a measurable, bounded function over a bounded set. By means of Fréchet's (Bulletin de la Société Mathématique de France, vol. 43) extension of the Borel-Lebesgue theorem, continuous functions of a certain type are shown to be integrable. Finally, in case the variable is n -dimensional real or in case the values of the function are real, our definition of an integral is shown to be equivalent to those of Bochner (Fundamenta Mathematicae, Vol. 20, 1933) and Saks (Saks: *Théorie de l'Intégrale*, Warsaw, 1933). (Received November 3, 1934.)

382. Mr. William A. Mersman: *A new type of abstract integral.*

The purpose of the paper is to develop an integral of abstract-valued functions of an abstract variable which will be less artificial in its fundamental definitions than any hitherto considered. This is done by means of a simple generalization of the definition of a measurable function as given by Titchmarsh (Titchmarsh: *The Theory of Functions*, Oxford, 1932). Although our definition is shown to be equivalent to those of Saks (Saks: *Théorie de*

l'Integrale, Warsaw, 1933) and Bochner (Bochner: *Fundamenta Mathematicae*, vol. 20, 1933) in the special cases which they consider, it permits simpler methods of proof. Some interesting results are obtained regarding functions of two variables, one being measurable and the other in a metric space. For functions of a single variable the integral is shown to be a linear, additive, continuous functional as well as a completely additive function of sets. Finally, the definition is extended so that any bounded, convergent series of integrable functions can be integrated term by term. (Received November 3, 1934.)

383. Professor A. D. Michal: *Abstract euclidean spaces and semi-vector spaces*. Preliminary report.

In an abstract Hilbert space the relation between the norm and the Hermitian form (f, g) is given by $\|f\| = (f, f)^{1/2}$. In this paper postulates are given for normed vector spaces with a continuous Hermitian form in which this relation need not be satisfied. Real as well as complex cases are considered. The second part of the paper deals with modified normed vector spaces for which the postulate $\|\alpha \odot f\| \leq |\alpha| \|f\|$ takes the place of $\|\alpha \odot f\| = |\alpha| \|f\|$. Semi-vectorial rings are also considered. An interesting instance is given by a commutative non-associative vectorial ring of Hermitian matrices. (Received November 3, 1934.)

384. Professor A. D. Michal: *Theorems on integral invariants of associated type*.

The Cartan complete integral invariants are special integral invariants of the associated type. In this paper the following theorem, among others, is proved: a necessary and sufficient condition that I_p^* be an associated integral invariant is that it be in the form $I_p^* = I_p^c + (I_{p-1}, \delta t)$ where I_p^c is a Cartan complete integral invariant and the term in parentheses denotes the symbolic integral product of a $p-1$ dimensional Poincaré integral invariant and δt . (Received November 3, 1934.)

385. Professor A. D. Michal: *A theorem on total Fréchet differentials in abstract vector spaces*.

In this paper the author proves the following theorem. If (a) $f(x, y_1, y_2, \dots, y_n)$ is linear (additive and continuous) in each y_i ; (b) the partial Fréchet differential $f_x(x, y_1, y_2, \dots, y_n; x)$ is continuous in x for x in $\|x - x_0\| < a$ and continuous separately in each y_i , then the total Fréchet differential of $f(x, y_1, \dots, y_n)$ exists and is continuous jointly in all its variables for x in $\|x - x_0\| < a$ and all $\delta x, y_i$, and δy_i . (Received November 3, 1934.)

386. Professor A. D. Michal and Mr. V. Elconin: *Solutions of total differential equations in abstract vector spaces as functions of the initial parameters*.

The authors continue their studies of completely integrable Pfaffian equations $\delta\varphi(x) = F(x, \varphi(x), \delta x)$ in Banach spaces (see this Bulletin, abstract 40-7-228). The solution function $\varphi(x, x_{0j}, \varphi_0)$ taking on the initial value φ_0 for

$x = x_0$ is shown to possess partial Fréchet differentials $\delta_x \phi(x, x_0, \phi_0; \delta x_0)$ and $\delta_{\phi_0} \varphi(x, x_0, \varphi_0; \delta \varphi_0)$. (Received November 3, 1934.)

387. Mr. Victor Elconin: *Multiple differentials and integrals of abstractly-valued functions.*

Multiple differentials and integrals of functions on an n -space to a Banach space are defined and their relations investigated. Conditions on the integrand are found, necessary and sufficient for the existence of the integral, and sufficient for its continuity and differentiability with respect to the limits of integration and the integrand parameters. The commutativity of multiple integral operators with multiple differential operators that act only on the integrand is established for continuous differentiable integrands. It is shown that a change in the order of repeated multiple integration leaves the integral unaltered if the integrand is continuous over the range of integration. Interesting relations between differentials and integrals of different multiplicities are found, including generalizations of theorems on differentials due to M. Fréchet, A. D. Michal, and M. Kerner. (Received November 3, 1934.)

388. Mr. A. E. Taylor: *A reduced set of postulates for Hilbert space.* Preliminary report.

The usual postulates for abstract Hilbert space contain several redundancies. A reduced set is obtained, and questions of consistency, independence, and categoricalness are examined. (Received November 3, 1934.)

389. Mr. A. E. Taylor: *Integral invariants and integrals of systems of first order differential equations.*

The use of a new type of integral invariant leads to results which are new. Especially interesting is the case of associated integral invariants of n th order, for then we get two multipliers, and hence an integral of the system. From this result, using a theorem of Koenigs, we deduce a second integral. (Received November 3, 1934.)

390. Mr. A. E. Taylor: *Additions to the theory of integral invariants.*

This paper is largely concerned with a new type of integral invariant, which I have called an *associated* invariant. It is an integral invariant, in the Poincaré sense, of an associated differential system, It bears an important relation to Cartan's *complete* integral invariant, which is an attached Poincaré integral invariant of the same associated system. Necessary and sufficient conditions are deduced, and various fundamental relationships are established. In the case of first order *relative* invariants, the theory is developed independently of *absolute* integral invariants, and the use of Stokes's Theorem in hyperspace is avoided. (Received November 3, 1934.)