THE PRINCIPAL MATRICES OF A RIEHMANN MATRIX*

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1. Introduction. A matrix \( \omega \) with \( p \) rows and \( 2p \) columns of complex elements is called a Riemann matrix if there exists a rational \( 2p \)-rowed skew-symmetric matrix \( C \) such that

\[
\omega C \omega' = 0, \quad \pi = i\omega C \bar{\omega}'
\]

is positive definite. The matrix \( C \) is called a principal matrix of \( \omega \) and it is important in algebraic geometry to know what are all principal matrices of \( \omega \) in terms of a given one. In the present note I shall solve this problem.

2. Principal Matrices. A rational \( 2p \)-rowed square matrix \( A \) is called a projectivity of \( \omega \) if

\[
\alpha \omega = \omega A
\]

for a \( p \)-rowed complex matrix \( \alpha \). The Riemann matrices \( \omega \) have recently† been completely classified in terms of their projectivities; so we may regard all the projectivities \( A \) of \( \omega \) as known.

A projectivity \( A \) is called symmetric if \( CA'C^{-1} = A \). Let \( A \) be a symmetric projectivity so that if \( B = AC \), then \( B'(AC)' = -CA' = -AC = -B \) is a skew-symmetric matrix. Then \( iAC \) is Hermitian and so must be

\[
\delta = \omega(iAC)\bar{\omega}' = \alpha(i\omega C \bar{\omega}') = \alpha \pi.
\]

Now \( \pi \) is positive definite so that \( \pi = \rho \rho' \), where \( \rho \) is non-singular. Then \( \pi^{-1} = (\rho')^{-1} \rho^{-1} = \bar{\sigma}' \sigma \) with \( \sigma \) non-singular. Hence \( \alpha = \delta \pi^{-1} = \delta \bar{\sigma}' \sigma \) and

\[
\sigma \alpha \sigma^{-1} = \sigma \bar{\omega}'.
\]

The matrix \( \sigma \bar{\omega}' \) is evidently Hermitian and it is well known that then \( \sigma \bar{\omega}' \) and the similar matrix \( \alpha \) have only simple ele-

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mentary divisors and all real characteristic roots. Thus \( \alpha = \beta \gamma \beta^{-1} \), where \( \gamma \) is a real diagonal matrix.

Write

\[
\Omega = \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix},
\]

so that, as is well known, and may easily be computed,

\[ A = \Omega^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \Omega = \Lambda \Gamma \Lambda^{-1}, \]

where

\[ \Gamma = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad \Lambda = \Omega^{-1} \begin{pmatrix} \beta & 0 \\ 0 & \overline{\beta} \end{pmatrix}. \]

Then \( A \) is similar to the real diagonal matrix \( \Gamma \) and we have proved the following theorem.*

**Theorem 1.** A symmetric projectivity of a Riemann matrix has all simple elementary divisors and all real characteristic roots.

We may now determine all principal matrices of a given Riemann matrix \( \omega \) with a given principal matrix \( C \). Let \( B \) be a second principal matrix of \( \omega \) so that \( \omega B \omega' = 0 \). It is well known that \( BC = A \) is a projectivity of \( \omega \). In fact \( \omega A = \omega A \), where \( \alpha = \delta \pi^{-1} \) is defined by (3). Moreover \( B' = -B \), so that

\[ (AC)' = C'A' = -CA' = -AC, \]

and \( CA'C^{-1} = A \). Hence \( A = BC^{-1} \) is a symmetric projectivity of \( \omega \).

The matrix \( \delta = i \omega B \overline{\omega} \) is positive definite if \( B \) is a principal matrix of \( \omega \). Hence \( \sigma \delta \overline{\tau} \) is positive definite and has all positive characteristic roots. The matrices \( \alpha \) and \( \gamma \) defined above are similar to \( \sigma \omega \sigma^{-1} = \sigma \delta \overline{\tau} \) and have the same characteristic roots, so that the diagonal matrix \( \Gamma \), whose diagonal elements are these characteristic roots repeated, has all positive diagonal elements. Then \( A \), which is similar to \( \Gamma \), has all positive characteristic roots.

Conversely, let \( A \) be a symmetric projectivity of \( \omega \) with all positive characteristic roots. Then \( \Gamma \) has all positive diagonal

* The proof by the use of (4) was suggested by certain analogous considerations of N. Jacobson.
elements, $\alpha$ has all positive characteristic roots and so has $\sigma \alpha^{-1} = \sigma \overline{\alpha}^\prime$. But $\sigma \overline{\alpha}^\prime$ is an Hermitian matrix with characteristic roots all positive. Then $\sigma \overline{\alpha}^\prime$ is positive definite and so is $\delta = i \omega A \omega^\prime$. Moreover, if $B = AC$, then

$$\omega B \omega^\prime = \omega A \omega^\prime = \alpha \omega C \omega^\prime = 0$$

and $B$ is a principal matrix of $\omega$. We have proved the following result.

**Theorem 2.** Let $\omega$ be a Riemann matrix with principal matrix $C$ and let $A$ range over the set of all symmetric projectivities of $\omega$ which have positive characteristic roots. Then a rational matrix $B$ is a principal matrix of $\omega$ if and only if $B = AC$ with $A$ in the above set.

3. **Pure Riemann Matrices of the First Kind.** The problem of determining what projectivities of $\omega$ are symmetric with all characteristic roots positive is, in general, a complicated one. We may nevertheless solve this problem for the case where $\omega$ is a pure Riemann matrix of the first kind.

The multiplication algebra of a pure Riemann matrix is a division algebra $D$. The centrum of $D$ is a field represented by a field $R(S)$ of all polynomials with rational coefficients of a projectivity $S$ of $\omega$. Algebra $D$ is of the first or second kind according as $S$ is symmetric.

If $D$ is of the first kind, then I have proved* that every projectivity of $\omega$ has the form $\varphi(S)$ in $R(S)$ or the form

(7) $$\alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 XY,$$

with $\alpha_1, \ldots, \alpha_4$ in $R(S)$, such that

(8) $$XY = -XY, \quad X^2 = \xi, \quad Y^2 = \eta, \quad (\xi, \eta \text{ in } R(S)).$$

The order of the set of all symmetric projectivities of $\omega$ is its singularity index $k$. If $S$ is symmetric and $R(S)$ has order $t$, then $k=t$ or $k=3t$ according as we may not or may take both $X$ and $Y$ symmetric, while $k=t$ if $D$ is equivalent to $R(S)$.

Let first $k=t$ so that every symmetric projectivity of $\omega$ is in $R(S)$, and let the characteristic roots of $S$ be $\sigma_1, \ldots, \sigma_t$. Then

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if $A = p(S)$, the characteristic roots of $A$ are $p(\sigma_i)$ and we have the following theorem.

**Theorem 3.** Let $\omega$ be a pure Riemann matrix of the first kind with projectivity algebra $D_0$ over $R(S)$ having singularity index $k = t$. Then the principal matrices of $\omega$ are the matrices

$$p(S)C,$$

where $p(S)$ is a polynomial in $S$ with rational coefficients such that

$$p(\sigma_j) > 0, \quad (j = 1, \cdots, t).$$

Next let $k = 3t$ so that every symmetric projectivity of $\omega$ has the form

$$A = p_1(S) + p_2(S)X + p_3(S)Y.$$  

Then $A$ satisfies the equation in an indeterminate $\alpha$

$$[\alpha - p_1(S)]^3 = [p_2(S)]^2\xi + [p_3(S)]^2\eta.$$  

Hence the characteristic roots of $A$ are the numbers

$$p_1(\sigma_i) \pm \{[p_2(\sigma_i)]^2\xi(\sigma_i) + [p_3(\sigma_i)]^2\eta(\sigma_i)\}^{1/2}.$$  

Since $X$ and $Y$ are symmetric we have the well known trivial result

$$\xi(\sigma_i) > 0, \quad \eta(\sigma_i) > 0.$$  

But then the characteristic roots of $A$ are all positive if and only if

$$p_1(\sigma_i) > \{[p_2(\sigma_i)]^2\xi(\sigma_i) + [p_3(\sigma_i)]^2\eta(\sigma_i)\}^{1/2}.$$  

We have proved the following theorem.

**Theorem 4.** Let $\omega$ be pure with singularity index $k = 3t$ and let $p_1(S), p_2(S), p_3(S)$ be polynomials in $S$ with rational coefficients. Then every principal matrix of $\omega$ is given by the set of matrices

$$[p_1(S) + p_2(S)X + p_3(S)Y]C,$$

with $p_1, p_2, p_3$ chosen so that (14) holds.