ON THE THEORY OF RESIDUES OF
POLYGENIC FUNCTIONS*

BY V. C. POOR

1. Introduction. In polygenic function theory we are interested
in a sub-class of the class of all functions, such that

\[ f(z) = f(\xi, \bar{\xi}), \]

where \( \xi \) and \( \bar{\xi} \) are complex variables; that is, such that when \( \xi \)
and \( \bar{\xi} \) are assigned \( f(z) \) is known. The particular sub-class to
which we restrict ourselves is the class such that \( \bar{\xi} \) is always the
conjugate of \( \xi \), or

\[ f(z) = f(z, \bar{z}). \]

For a brief outline of this subject and a quite complete bibliog­
raphy one should consult the paper by Hedrick\( ^\dagger \) in this Bulletin.

It is the purpose of this paper to generalize the definitions for
residues of polygenic functions previously given\( ^\ddagger \) and to extend
the theory. Incidentally in the process, the circulation theo­
rems\( ^\S \) are generalized; a theorem on residues of regular functions
is obtained, while the theory is applied to the large class of func­
tions defined by a Laurent series.

2. The Definitions for Residues. If we surround the point \( z = a \)
by a circle \( O \), center at \( a \) and radius \( r \), then the residue \( R_z \) of
\( f(z) \) is defined by the equation,\( ||\)

\[ R_z = \lim_{r \to 0} \frac{1}{2\pi i} \int_{O} f(z) \, dz; \]

while the residue \( R_{\bar{z}} \), which is of equal importance, is

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* Presented to the Society, December 27, 1933.
|| Poor, loc. cit., §1.

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\[ R_s = \lim_{r \to 0} -\frac{1}{2\pi i} \int_{\partial D} f(z) dz; \]

the total residue is then, by definition,

\[ R = R_s + R_{\neq}. \]

**Theorem 1.** In the definitions for the residues of a polygenic function, the contour of integration may be replaced by any arbitrary contour \( C \), providing in the limit the area bounded by \( C \) contracts to the point \( a \).

In this theorem we are interested in a point \( a \) of a domain \( D \) at which \( R_s \) exists. We wish to prove then, when the circle of radius \( r \) is replaced by an arbitrary contour \( C \), that

\[ \lim_{C \to 0} \frac{1}{2\pi i} \int_{\partial \Gamma} f(z) dz \]

exists and is equal to \( R_s \) if \( C \) is contracted to the point \( z = a \).

We surround the point \( a \) by two concentric circles, \( O_1 \) and \( O_2 \), with centers at \( a \), and with radii \( r_1 \) and \( r_2 \), respectively. We also enclose \( a \) by an arbitrary contour \( C \) of length \( l \) such that

\[ 2\pi r_2 \leq l \leq 2\pi r_1, \]

with the condition that the contour \( C \) lie between the two circles and that it contract in the limit to a point. The contour \( C \), then, divides the area, \( \sigma \), between the two circles into two annular parts, \( \sigma_1 \) and \( \sigma_2 \), adjacent to circles \( O_1 \) and \( O_2 \), respectively, so that \( \sigma_1 + \sigma_2 = \sigma \).

We make the proof of this theorem depend on the following lemma.

**Lemma.** The circulation theorem,

\[ 2i \int_{\partial \Gamma} \frac{\partial f}{\partial \bar{z}} d\sigma = \int_{C} f(z) dz, \]

is extensible to an annular domain bounded by two closed contours.

Let \( C_1 \) be a closed contour lying within the area bounded by the closed contour \( C_2 \). Let \( C_3 \) be a closed contour in the annular domain determined by \( C_1 \) and \( C_2 \) and not inclosing \( C_1 \). We assume that the circulation theorem (4) is valid for every such
contour \( C_3 \) in the annular domain. We now join a point \( P_2 \) of \( C_2 \) with a point \( P_1 \) of \( C_1 \) without intersecting the curve \( C_3 \). We then distort the contour \( C_3 \) till it coincides with \( C_1, C_2 \) and the curve joining the two points. When we take the flow around this distorted contour, possibly keeping the area to the left, the line integrals from \( P_2 \) to \( P_1 \) and from \( P_1 \) to \( P_2 \) just cancel each other. And this evidently establishes the lemma.

In passing we may mention that this lemma is a simple generalization of the usual extension of the Cauchy first law. Also the circulation theorem

\[
(4') \quad -2i \int_{\sigma} \frac{\partial f}{\partial \overline{z}} \, d\sigma = \int_{\gamma} f(z) \, d\overline{z},
\]

involving the Kasner mean derivative, may be extended in a similar way.

The application of the lemma to the area \( \sigma \) between \( O_1 \) and \( O_2 \) gives

\[
(5) \quad \int_{o_2} f(z) \, dz - \int_{o_1} f(z) \, dz = 2i \int_{\sigma} \frac{\partial f}{\partial \overline{z}} \, d\sigma,
\]

where the line integrals are both taken counter-clockwise around the point \( a \). But the paths of integration are circles; the line integrals in (5) therefore have the same limits.

Thus

\[
(6) \quad \text{limit}_{r_1 \to 0} \int_{\sigma} \frac{\partial f}{\partial \overline{z}} \, d\sigma = 0.
\]

When we apply the lemma to the areas \( \sigma_2 \) and \( \sigma_1 \), taking the line integrals counter-clockwise around \( a \), as before, we may write

\[
(7) \quad \int_{o_2} f(z) \, dz - \int_{\sigma_2} f(z) \, dz = 2i \int_{\sigma_2} \frac{\partial f}{\partial \overline{z}} \, d\sigma,
\]

and

\[
(8) \quad \int_{\sigma} f(z) \, dz - \int_{o_1} f(z) \, dz = 2i \int_{\sigma_1} \frac{\partial f}{\partial \overline{z}} \, d\sigma.
\]

If we subtract (8) from (7) we will find that
If \( f(z) \) is analytic at \( z = a \), \( \partial f / \partial \bar{z} \) will vanish in the limit, so that the right member of (9) becomes zero. The theorem is thus verified for this case. Also if \( f(z) \) is regular at \( z = a \), then \( \partial f / \partial \bar{z} \) is unique and bounded at \( z = a \). Thus for \( \sigma_{1,2} \), that is, for \( \sigma_1 \) or \( \sigma_2 \), we have, if \( M \) is the maximum value of \( |\partial f / \partial \bar{z}| \) in \( \sigma_{1,2} \),

\[
\left| \int_{\sigma_{1,2}} \frac{\partial f}{\partial \bar{z}} \, d\sigma \right| \leq \int_{\sigma_{1,2}} \left| \frac{\partial f}{\partial \bar{z}} \right| \, d\sigma \leq M_{\sigma_{1,2}},
\]

which goes to zero with \( \sigma_{1,2} \). Thus the right member of (9) is zero in the limit and the theorem is again validated.

If \( f(z) \) is neither analytic nor regular at \( a \), then (6) is satisfied by the ultimate cancellation of the elements of the integrand or because of the rapidity with which the area \( \sigma \) goes to zero. It would appear to be a very special function which would satisfy the first condition, while (6) is evidently valid when

\[
\lim_{r \to 0} \int_{\sigma} \left| \frac{\partial f}{\partial \bar{z}} \right| \, d\sigma = 0.
\]

For such functions, then,

\[
\left| \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}} \, d\sigma - \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}} \, d\sigma \right| \leq \int_{\sigma} \left| \frac{\partial f}{\partial \bar{z}} \right| \, d\sigma,
\]

so that in the limit the left member of (9) vanishes. We have thus proved the theorem under the restriction imposed by (10).

As a simple example we might take \( 1/z + \bar{z} \) as \( f(z) \) with the origin as the point \( a \). Here \( \partial f / \partial \bar{z} = 1 \), so that

\[
\int_{\sigma} \frac{\partial f}{\partial \bar{z}} \, d\sigma = \int_{\sigma} d\sigma = \pi (r_2^2 - r_1^2),
\]

which evidently goes to zero with \( r_2 \). We have

\[
\int_{0_z} (\frac{1}{z} + \bar{z}) \, dz = 2\pi i + 2\pi i r_2^2,
\]
which in the limit is $2\pi i$. Thus, by the theorem,

$$\lim_{c \to 0} \int_C \left( \frac{1}{z} + \bar{z} \right) dz = 2\pi i,$$

and this result can easily be verified. From analytic function theory $\int_0 dz/z = 2\pi i$, while if we use $r$ for the maximum value of $\bar{z}$, we have

$$\left| \int_C \bar{z} dz \right| \leq \int_C |\bar{z}| |dz| \leq rl \leq 2\pi r^2,$$

which goes to zero with $r^2$.

3. Regular Functions. We define a function to be regular in a domain $D$ of the complex plane when it possesses a differential at every point of the domain. If $f(z)$ has a differential at every point $z = a$ of the domain, then

$$f(z) - f(a) \equiv \frac{\partial f}{\partial z} (z - a) + \frac{\partial f}{\partial \bar{z}} (\bar{z} - \bar{a}) + \eta,$$

where $\lim_{z \to a} \eta = 0$.

**Theorem 2.** The residue of every polygenic function regular in a domain $D$ is identically zero at every point of the domain.

In evaluating the integral in (1), we have

$$\int_0 f(z) dz = \int_0 f(a) dz + \int_0 [f(z) - f(a)] dz$$

$$= f(a) \int_0 dz + \frac{\partial f}{\partial z} \int_0 (z - a) dz$$

$$+ \frac{\partial f}{\partial \bar{z}} \int_0 (\bar{z} - \bar{a}) dz + \int_0 \eta dz.$$

The first two integrals in this last member vanish identically since the integrands are analytic functions, while each of the last two integrals contains a factor $r^1 + \varepsilon$, where $0 < \varepsilon \leq 1$; thus these integrals go to zero with $r$. Hence $R_z = 0$.

In a similar way we can show that $R_{\bar{z}}$ is also zero. Hence $R = 0$ at every point of the domain.
4. Functions Defined by a Laurent Series. In this section we shall be interested in the residues of a class of polygenic functions defined by a Laurent series. It will be sufficiently general to study the series in the neighborhood of the origin. Thus let

\begin{equation}
(12) \quad f(z) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{a_{n-k,k}}{z^{n-k} \bar{z}^k} + a_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} b_{n-k,k} z^{n-k} \bar{z}^k;
\end{equation}

we proceed to find the residue $R_z$ of $f(z)$ according to definition (1). The contribution to $R_z$ due to the term whose denominator is $z^{n-k} \bar{z}^k$, $(n \neq 1)$, will contain the factor

\[ \int_0^{2\pi} \frac{dz}{z^{n-k} \bar{z}^k}. \]

For $z$ we write $z = re^{i\theta}$; then $\bar{z} = re^{-i\theta}$ and $dz = rie^{i\theta} \, d\theta$, so that we have

\[ \int_0^{2\pi} \frac{dz}{z^{n-k} \bar{z}^k} = \int_0^{2\pi} \frac{r \, i e^{i\theta} \, d\theta}{r^{n-k} e^{i(n-2k)\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1+2k-n)\theta} \, d\theta = 0, \]

since $1 + 2k - n$ is a positive or negative integer. Similarly the contribution to $R_z$ by the term $b_{n-k,k} z^{n-k} \bar{z}^k$ will contain the factor $\int_0^{2\pi} z^{n-k} \bar{z}^k \, dz$. Hence

\[ \int_0 z^{n-k} \bar{z}^k \, dz = \int_0 i r^{n+1} e^{i(n-2k+1)\theta} \, d\theta = i r^{n+1} \cdot 0 = 0. \]

Since the residue of $a_0$ is zero, the only contribution to $R_z$ will come from the term involving $a_{1,0}$ as a factor. Therefore

\[ R_z = \lim_{r \to 0} \frac{a_{1,0}}{2\pi i} \int_0^{2\pi} \frac{dz}{z} = \frac{a_{1,0}}{2\pi i} \int_0^{2\pi} i \, d\theta = \frac{a_{1,0}}{2\pi i} \cdot 2\pi i = a_{1,0}. \]

The same procedure will convince one that the only contribution to $R_z$ will be furnished by the term involving the factor $a_{0,1}$. Then by definition

\[ R_z = \lim_{r \to 0} \frac{-1}{2\pi i} \int_0 \frac{a_{0,1} \, d\bar{z}}{\bar{z}} = \lim_{r \to 0} \frac{ia_{0,1}}{2\pi i} \int_0^{2\pi} d\theta = \frac{2\pi i a_{0,1}}{2\pi i} = a_{0,1}. \]

Hence the total residue at the origin of a function defined by a Laurent series is
A MERSENNE NUMBER

BY R. E. POWERS

I have recently determined by the computation of Lucas' series 4, 14, 194, · · · * that the number \( N = 2^{241} - 1 \) is composite, since the 240th term of the series is congruent to

\[
- 98 \ 6778335538 \ 8807227981 \ 3604528486 \ 9326522489 \ 7467133466 \\
0099172867 \ 1619979800 \ (\text{mod} \ N).
\]

This term would be zero if \( N \) were prime.

The square of each term was obtained by means of a computing machine, D. N. Lehmer's cross-multiplication† being used; and these squares, diminished by 2, were divided by \( N \) by hand, with the aid of a table of the 1000 multiples of \( N \): \( N, 2N, 3N, \cdots, 1000N \), the quotients being thus obtained three or more digits at a time, and the computation was checked throughout by the four moduli 9, \( 10^3 + 1 \), \( 10^4 + 1 \), and \( 10^7 + 1 \).

DENVER, COLORADO