ON THE THEORY OF RESIDUES OF POLYGENIC FUNCTIONS*

BY V. C. POOR

1. *Introduction.* In polygenic function theory we are interested in a sub-class of the class of all functions, such that

$$f(z) = f(\xi, \zeta),$$

where ξ and ζ are complex variables; that is, such that when ξ and ζ are assigned $f(z)$ is known. The particular sub-class to which we restrict ourselves is the class such that ζ is always the conjugate of ξ, or

$$f(z) = f(z, \bar{z}).$$

For a brief outline of this subject and a quite complete bibliography one should consult the paper by Hedrick† in this Bulletin.

It is the purpose of this paper to generalize the definitions for residues of polygenic functions previously given‡ and to extend the theory. Incidentally in the process, the circulation theorems§ are generalized; a theorem on residues of regular functions is obtained, while the theory is applied to the large class of functions defined by a Laurent series.

2. *The Definitions for Residues.* If we surround the point $z=a$ by a circle O, center at a and radius r, then the residue R_z of $f(z)$ is defined by the equation,‖

$$R_z = \lim_{r \to 0} \frac{1}{2\pi i} \int_O f(z) dz; \tag{1}$$

while the residue $R_{\bar{z}}$, which is of equal importance, is

* Presented to the Society, December 27, 1933.

† E. R. Hedrick, *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933), pp. 75–96.

‡ V. C. Poor, *Residues of polygenic functions*, Transactions of this Society, vol. 32 (1930), pp. 216–222.

§ Poor, loc. cit. Calugaréano, (Thesis), *Sur les fonctions polygènes d'une variable complexe*, 1928, p. 11.

‖ Poor, loc. cit., §1.

$$R_{\bar{z}} = \underset{r\to 0}{\text{limit}} \frac{-1}{2\pi i} \int_O f(z) d\bar{z}; \tag{2}$$

the total residue is then, by definition,

$$R = R_z + R_{\bar{z}}. \tag{3}$$

THEOREM 1. *In the definitions for the residues of a polygenic function, the contour of integration may be replaced by any arbitrary contour C, providing in the limit the area bounded by C contracts to the point a.*

In this theorem we are interested in a point a of a domain D at which R_z exists. We wish to prove then, when the circle of radius r is replaced by an arbitrary contour C, that

$$\underset{C\to 0}{\text{limit}} \frac{1}{2\pi i} \int_C f(z) dz$$

exists and is equal to R_z if C is contracted to the point $z=a$.

We surround the point a by two concentric circles, O_1 and O_2, with centers at a, and with radii r_1 and r_2, respectively. We also enclose a by an arbitrary contour C of length l such that

$$2\pi r_2 \geqq l \geqq 2\pi r_1,$$

with the condition that the contour C lie between the two circles and that it contract in the limit to a point. The contour C, then, divides the area, σ, between the two circles into two annular parts, σ_1, and σ_2, adjacent to circles O_1 and O_2, respectively, so that $\sigma_1+\sigma_2=\sigma$.

We make the proof of this theorem depend on the following lemma.

LEMMA. *The circulation theorem,*

$$2i \int_\sigma \frac{\partial f}{\partial \bar{z}} d\sigma = \int_C f(z) dz, \tag{4}$$

is extensible to an annular domain bounded by two closed contours.

Let C_1 be a closed contour lying within the area bounded by the closed contour C_2. Let C_3 be a closed contour in the annular domain determined by C_1 and C_2 and not inclosing C_1. We assume that the circulation theorem (4) is valid for every such contour C_3 in the annular domain. We now join a point P_2 of C_2 with a point P_1 of C_1 without intersecting the curve C_3. We then distort the contour C_3 till it coincides with C_1, C_2 and the curve joining the two points. When we take the flow around this distorted contour, possibly keeping the area to the left, the line integrals from P_2 to P_1 and from P_1 to P_2 just cancel each other. And this evidently establishes the lemma.

In passing we may mention that this lemma is a simple generalization of the usual extension of the Cauchy first law. Also the circulation theorem

$$-2i\int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma = \int_C f(z)d\bar{z}, \tag{4'}$$

involving the Kasner mean derivative, may be extended in a similar way.

The application of the lemma to the area σ between O_1 and O_2 gives

$$\int_{O_2} f(z)dz - \int_{O_1} f(z)dz = 2i\int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma, \tag{5}$$

where the line integrals are both taken counter-clockwise around the point a. But the paths of integration are circles; the line integrals in (5) therefore have the same limits.

Thus

$$\operatorname*{limit}_{r_2 \to 0} \int_\sigma \frac{\partial f}{\partial \bar{z}}\,d\sigma \equiv 0. \tag{6}$$

When we apply the lemma to the areas σ_2 and σ_1, taking the line integrals counter-clockwise around a, as before, we may write

$$\int_{O_2} f(z)dz - \int_C f(z)dz = 2i\int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\,d\sigma, \tag{7}$$

and

$$\int_C f(z)dz - \int_{O_1} f(z)dz = 2i\int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\,d\sigma. \tag{8}$$

If we subtract (8) from (7) we will find that

$$
\begin{aligned}
&\int_{O_2} f(z)dz + \int_{O_1} f(z)dz - 2\int_C f(z)dz \\
(9) \qquad &= 2i\left[\int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\, d\sigma - \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\, d\sigma\right].
\end{aligned}
$$

If $f(z)$ is analytic at $z=a$, $\partial f/\partial \bar{z}$ will vanish in the limit, so that the right member of (9) becomes zero. The theorem is thus verified for this case. Also if $f(z)$ is regular at $z=a$, then $\partial f/\partial \bar{z}$ is unique and bounded at $z=a$. Thus for $\sigma_{1,2}$, that is, for σ_1 or σ_2, we have, if M is the maximum value of $|\partial f/\partial \bar{z}|$ in $\sigma_{1,2}$,

$$
\left| \int_{\sigma_{1,2}} \frac{\partial f}{\partial \bar{z}}\, d\sigma \right| \leqq \int_{\sigma_{1,2}} \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma \leqq M\sigma_{1,2},
$$

which goes to zero with $\sigma_{1,2}$. Thus the right member of (9) is zero in the limit and the theorem is again validated.

If $f(z)$ is neither analytic nor regular at a, then (6) is satisfied by the ultimate cancellation of the elements of the integrand or because of the rapidity with which the area σ goes to zero. It would appear to be a very special function which would satisfy the first condition, while (6) is evidently valid when

$$
(10) \qquad \operatorname*{limit}_{r_2 \to 0} \int_{\sigma} \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma = 0.
$$

For such functions, then,

$$
\left| \int_{\sigma_2} \frac{\partial f}{\partial \bar{z}}\, d\sigma - \int_{\sigma_1} \frac{\partial f}{\partial \bar{z}}\, d\sigma \right| \leqq \int_{\sigma} \left| \frac{\partial f}{\partial \bar{z}} \right| d\sigma,
$$

so that in the limit the left member of (9) vanishes. We have thus proved the theorem under the restriction imposed by (10).

As a simple example we might take $1/z+\bar{z}$ as $f(z)$ with the origin as the point a. Here $\partial f/\partial \bar{z}=1$, so that

$$
\int_{\sigma} \frac{\partial f}{\partial \bar{z}}\, d\sigma = \int_{\sigma} d\sigma = \pi(r_2^2 - r_1^2),
$$

which evidently goes to zero with r_2. We have

$$
\int_{O_2^{\text{Q}}} \left(\frac{1}{z} + \bar{z}\right) dz = 2\pi i + 2\pi i r_2^2,
$$

which in the limit is $2\pi i$. Thus, by the theorem,

$$\operatorname*{limit}_{C\to 0} \int_C \left(\frac{1}{z} + \bar{z}\right) dz = 2\pi i,$$

and this result can easily be verified. From analytic function theory $\int_C dz/z = 2\pi i$, while if we use r for the maximum value of $\bar{z}$, we have

$$\left| \int_C \bar{z} dz \right| \leqq \int_C |\bar{z}| \, |dz| \leqq rl \leqq 2\pi r_2^2,$$

which goes to zero with r_2.

3. *Regular Functions.* We define a function to be regular in a domain D of the complex plane when it possesses a differential at every point of the domain. If $f(z)$ has a differential at every point $z = a$ of the domain, then

$$f(z) - f(a) \equiv \frac{\partial f}{\partial z}(z - a) + \frac{\partial f}{\partial \bar{z}}(\bar{z} - \bar{a}) + \eta, \tag{11}$$

where $\text{limit}_{z\to a}\eta = 0$.

THEOREM 2. *The residue of every polygenic function regular in a domain D is identically zero at every point of the domain.*

In evaluating the integral in (1), we have

$$\begin{aligned} \int_O f(z)dz &\equiv \int_O f(a)dz + \int_O [f(z) - f(a)]dz \\ &= f(a)\int_O dz + \frac{\partial f}{\partial z}\int_O (z - a)dz \\ &\quad + \frac{\partial f}{\partial \bar{z}}\int_O (\bar{z} - \bar{a})dz + \int_O \eta dz. \end{aligned}$$

The first two integrals in this last member vanish identically since the integrands are analytic functions, while each of the last two integrals contains a factor $r^{1+\epsilon}$, where $0 < \epsilon \leqq 1$; thus these integrals go to zero with r. Hence $R_z \equiv 0$.

In a similar way we can show that $R_{\bar{z}}$ is also zero. Hence $R \equiv 0$ at every point of the domain.

4. *Functions Defined by a Laurent Series.* In this section we shall be interested in the residues of a class of polygenic functions defined by a Laurent series. It will be sufficiently general to study the series in the neighborhood of the origin. Thus let

$$f(z) \equiv \sum_{n=1}^{m}\sum_{k=0}^{n} \frac{a_{n-k,k}}{z^{n-k}\bar{z}^{k}} + a_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} b_{n-k,k}z^{n-k}\bar{z}^{k}; \tag{12}$$

we proceed to find the residue R_z of $f(z)$ according to definition (1). The contribution to R_z due to the term whose denominator is $z^{n-k}\bar{z}^{-k}$, $(n\neq 1)$, will contain the factor

$$\int_O \frac{dz}{z^{n-k}\bar{z}^{k}}.$$

For z we write $z=re^{i\theta}$; then $\bar{z}=re^{-i\theta}$ and $dz=rie^{i\theta}\,d\theta$, so that we have

$$\int_O \frac{dz}{z^{n-k}\bar{z}^{k}} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{r^n l^{i(n-2k)\theta}} = \frac{i}{r^{n-1}}\int_0^{2\pi} e^{i(1+2k-n)\theta}d\theta \equiv 0,$$

since $1+2k-n$ is a positive or negative integer. Similarly the contribution to R_z by the term $b_{n-k,k}\, z^{n-k}\bar{z}^k$ will contain the factor $\int_O z^{n-k}\,\bar{z}^k dz$. Hence

$$\int_O z^{n-k}\bar{z}^k dz = \int_O ir^{n+1}l^{i(n-2k+1)}d\theta = ir^{n+1}\cdot 0 = 0.$$

Since the residue of a_0 is zero, the only contribution to R_z will come from the term involving $a_{1,0}$ as a factor. Therefore

$$R_z = \operatorname*{limit}_{r\to 0} \frac{a_{1,0}}{2\pi i}\int_O \frac{dz}{z} = \frac{a_{1,0}}{2\pi i}\int_0^{2\pi} id\theta = \frac{a_{1,0}}{2\pi i}\cdot 2\pi i = a_{1,0}.$$

The same procedure will convince one that the only contribution to $R_{\bar{z}}$ will be furnished by the term involving the factor $a_{0,1}$. Then by definition

$$R_{\bar{z}} = \operatorname*{limit}_{r\to 0} \frac{-1}{2\pi i}\int_O \frac{a_{0,1}d\bar{z}}{\bar{z}} = \operatorname*{limit}_{r\to 0} \frac{ia_{0,1}}{2\pi i}\int_0^{2\pi} d\theta = \frac{2\pi i a_{0,1}}{2\pi i} = a_{0,1}.$$

Hence the total residue at the origin of a function defined by a Laurent series is

$$R = R_z + R_{\bar{z}} = a_{1,0} + a_{0,1}.$$

In conclusion it should be stated that $R_{\bar{z}}$ is the negative of $\bar{R}$ defined in the article in the Transactions (loc. cit.); this change is also carried over into the definition for the total residue. The reason for this change is partly evident in the results just obtained; then this change of sign brings $R_{\bar{z}}$ into accord with the mean derivative and the circulation theorem (4′). Also a slight change in the proof of Theorem 1 for R_z will establish the theorem for $R_{\bar{z}}$.

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NOTE ON A MERSENNE NUMBER

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I have recently determined by the computation of Lucas' series 4, 14, 194, $\cdots$ *that the number $N = 2^{241} - 1$ is composite, since the 240th term of the series is congruent to

$$-\ 98\ 6778335538\ 8807227981\ 3604528486\ 9326522489\ 7467133466\ 0099172867\ 1619979800 \pmod{N}.$$

This term would be zero if N were prime.

The square of each term was obtained by means of a computing machine, D. N. Lehmer's *cross-multiplication*† being used; and these squares, diminished by 2, were divided by N by hand, with the aid of a table of the 1000 multiples of N: N, $2N$, $3N$, $\cdots$, $1000N$, the quotients being thus obtained three or more digits at a time, and the computation was checked throughout by the four moduli 9, 10^3+1, 10^4+1, and 10^7+1.

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* This Bulletin, vol. 38 (1932), p. 383.

† American Mathematical Monthly, vol. 30 (1923), p. 67, and vol. 33 (1926), p. 199.