SOME OBSERVATIONS ON THE THEORY OF FOURIER TRANSFORMS

BY E. HILLE, A. C. OFFORD, AND J. D. TAMARKIN†

1. *From a Letter by Offord.* On pages 770–771 of your paper,‡ *On the theory of Fourier transforms*, you prove the following result.

LEMMA. *If $g(t)$ is in L_p, $1<p<\infty$, and if*

$$g(s, a) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin a(s-t)}{s-t}\,dt,$$

then

$$\lim_{u\to\infty}\int_{-\infty}^{\infty} | g(s) - g(s, a) |^p ds = 0.$$

You make no use of your hypothesis $p \leqq 2$ in this part of your paper. Now consider Berry's paper, *The Fourier transform identity theorem.*§

Write

$$G(s, a) = (2\pi)^{-1/2}\int_{-a}^{a} g(t)e^{-ist}dt,$$

and suppose $g(t)$ has a Fourier transform $G(s)$ in L_q, $1<q<\infty$, that is,

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} | G(s, a) - G(s) |^q ds = 0.$$

THEOREM A (Berry). *If $g(s)$ has a Fourier transform $G(s)$ in L_q, $1<q<\infty$, and if $G(s)$ has a Fourier transform $\mathfrak{G}(s)$ in L_p, $1<p<\infty$, then $\mathfrak{G}(s)=g(-s)$.*

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† The present note contains excerpts from a letter by Offord to Tamarkin, and from a reply to this letter by Hille and Tamarkin. Before knowing the contents of Offord's letter Hille arrived independently at some of Offord's conclusions, as well as at extensions in other directions.

‡ E. Hille and J. D. Tamarkin, this Bulletin, vol. 39 (1933), pp. 768–774.

§ Annals of Mathematics, (2), vol. 32 (1931), pp. 227–232.

This is a uniqueness theorem. The methods of your paper seem to give the existence theorem.

THEOREM B. *If $g(s)$ belongs to L_p, $1<p<\infty$, and if it has a Fourier transform $G(s)$ in some L_q, $1<q<\infty$, then $g(-s)$ is the Fourier transform in L_p of $G(s)$.*

We have

$$(*)\qquad G(s) = (2\pi)^{-1/2}\,\underset{a\to\infty}{\text{l.i.m.}}\int_{-a}^{a} g(t)e^{-ist}dt \qquad (\text{in } L_q).$$

Hence

$$\begin{aligned}(2\pi)^{-1/2}\int_{-b}^{b} G(u)e^{isu}du &= (2\pi)^{-1}\lim_{a\to\infty}\int_{-b}^{b} e^{isu}du\int_{-a}^{a} g(t)e^{-iut}dt \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} g(t)dt\int_{-b}^{b} e^{iu(s-t)}du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin b(s-t)}{s-t}\,dt.\end{aligned}$$

The desired result now follows from your lemma.

It is possible to go a little further than this.

THEOREM C. *If $g(s)$ belongs to L_p, $1<p<\infty$, and if*

$$\int_{-\infty}^{\infty} |G(s, a)|^q ds \leqq M, \qquad (1 < q < \infty),$$

for all a, then $g(s)$ has a Fourier transform $G(s)$ in L_q and $g(-s)$ is the Fourier transform in L_p of $G(s)$.

This follows from Theorems 1 and 7 of my paper, *On Fourier transforms*, III.† Theorem C is a corollary of the following theorem.

THEOREM D.‡ *Let $f(u)$ be integrable over every finite range and let*

$$F(x, a) = (2\pi)^{-1/2}\int_{-a}^{a} f(u)e^{-ixu}du.$$

† To appear in the Transactions of this Society.

‡ For the corresponding theorem when $p=\infty$ see A. C. Offord, *On Fourier transforms*, Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 197–216.

Let

$$(**)\qquad \int_{-\infty}^{\infty} |F(x, a)|^p dx \leqq M^p,$$

where $1 < p < \infty$. *Then* $F(x, a)$ *converges* $(C, 1)$ *in* L_p *to a function* $F(x)$ *and*

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u) e^{ixu} du$$

almost everywhere.

It is not assumed that $f(x)$ belongs to a Lebesgue class over $(-\infty, \infty)$. Let us write

$$F_1(x, a) = (2\pi)^{-1/2} \int_{-a}^{a} \left(1 - \frac{|u|}{a}\right) f(u) e^{-ixu} du.$$

If we can show that (**) implies

$$(***)\qquad \int_{-\infty}^{\infty} |F_1(x, a)|^p dx \leqq M^p,$$

then Theorem D will follow from Theorems 1 and 7 of my paper. Now

$$F_1(x, A) = A^{-1} \int_0^A F(x, a) da.$$

The desired result (***) follows from this by an easy application of Hölder's inequality.

2. *From an Answer by Hille and Tamarkin.*† The fact that our lemma above holds for any $p > 1$, and not only for $1 < p \leqq 2$, was observed by ourselves,‡ and also was called to our attention by

† It should be observed that results essentially equivalent to our Theorems 1, 3, 5 of the present note are contained in Offord's papers referred to above, although Offord does not use the notions of Fourier transforms in the sense of our Definitions 1 and 2. The paper of Offord mentioned in the preceding footnote was not available to us at the time when the present reply was being written.

‡ *A remark on Fourier transforms and functions analytic in a half plane*, Compositio Mathematica, vol. 1 (1934), pp. 98–102, especially p. 100. It should be stated that a result analogous to our lemma was obtained at a much earlier date by A. Berry, in an unpublished paper of his.

A. Zygmund. It is obvious that the proof of Theorem B does not utilize the full force of the condition (*). The only property of $g(s)$ which comes into play is that

$$\int_{-b}^{b} G(u)e^{isu}du = \lim_{a\to\infty} (2\pi)^{-1/2}\int_{-b}^{b} e^{isu}du \int_{-a}^{a} g(t)e^{-iut}dt$$
$$= \lim_{a\to\infty} \int_{-b}^{b} g(u, a)e^{isu}du$$

for all $b>0$ and all real s. This observation leads naturally to the following definition.

DEFINITION 1. *Let $f(x)$ be integrable over every finite range. Let*

$$f(x, a) = (2\pi)^{-1/2}\int_{-a}^{a} f(t)e^{-itx}dt. \tag{1}$$

If there exists a sequence $\{a_n\}$, $a_n \uparrow \infty$ such that, for all $b>0$ and all real x,

$$\lim_{n\to\infty} \int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} F(u)e^{ixu}du, \tag{2}$$

where $F(x)$ is integrable over every finite range, then $F(x)$ is called a Fourier transform of $f(x)$.

This definition finds its justification in the following theorem.

THEOREM 1. *If $f(x) \subset L_p$, $1 \leqq p < \infty$, and if $f(x)$ has a Fourier transform $F(x)$, then $F(x)$ is uniquely determined by $f(x)$, that is, does not depend on the choice of the particular sequence $\{a_n\}$ for which (2) is satisfied. If $p>1$, then $f(-x)$ is the Fourier transform in L_p of $F(x)$.*

Indeed from (2) and (1) we have

$$(2\pi)^{-1/2}\int_{-b}^{b} F(u)e^{ixu}du = (2\pi)^{-1}\lim_{n\to\infty}\int_{-b}^{b} e^{ixu}du\int_{-a_n}^{a_n} f(t)e^{-itu}dt$$
$$= (2\pi)^{-1}\lim_{n\to\infty}\int_{-a_n}^{a_n} f(t)dt\int_{-b}^{b} e^{iu(x-t)}du$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{\sin b(x-t)}{x-t}\,dt.$$

The right-hand member here does not depend on $\{a_n\}$. Hence, if $F_1(x)$ is any other Fourier transform of $f(x)$ in the sense of Definition 1, then again for all $b>0$ and all x,

$$(2\pi)^{-1/2}\int_{-b}^{b} F_1(u)e^{ixu}du = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{\sin b(x-t)}{x-t}\,dt,$$

which implies $F(x)=F_1(x)$, except perhaps for a set of values of x of measure zero. Consequently $F(x)$ is determined uniquely by $f(x)$, if it exists at all. The fact that $f(-x)$ is the Fourier transform in L_p of $F(x)$ if $p>1$ follows immediately from our Lemma.

Condition (2) is obviously satisfied if the set of functions $f(x, a)$, $a\geqq 0$, belongs to some L_q, $1<q<\infty$, and if the sequence $\{a_n\}$ can be selected in such a way that the sequence of functions $f(x, a_n)$ converges weakly in L_q to a weak limit $F(x)$. This will certainly be the case when the set $f(x, a)$, $a\geqq 0$, is bounded in L_q, that is, when

$$\int_{-\infty}^{\infty} |\, f(x, a)\,|^q \leqq M^q.$$

If this condition is satisfied, it is readily seen that there exists a function $F(x)\subset L_q$ such that $f(x, a)$ converges weakly to $F(x)$ as $a\to\infty$. This function $F(x)$ therefore appears as the Fourier transform of $f(x)$, while $f(-x)$ is the Fourier transform in L_p of $F(x)$. Your Theorem D furnishes a sharper result, however, namely, that $f(x, a)$ converges (strongly) to $F(x)$ in L_q and this even without the assumption that $f(x)\subset L_p$. In other words, the boundedness in L_q of the set $f(x, a)$ implies its convergence in L_q to the Fourier transform $F(x)$ of $f(x)$, and vice versa. The last statement is obvious.

Upon setting $\Phi(x)=\int_0^x F(t)dt$ we may rewrite (2) in the form

$$\lim_{n\to\infty}\int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} e^{ixu}d\Phi(u).$$

This immediately leads to a further generalization of the notion of the Fourier transform.

DEFINITION 2. *Let $f(x)$ be integrable over every finite range. Let $\Phi(x)$ be defined and finite for all values of x and integrable over* *every finite range. Let* $\int_\alpha^\beta e^{ixu}d\Phi(u)$ *be the generalized Stieltjes integral defined by*

$$(3)\qquad \int_\alpha^\beta e^{ixu}d\Phi(u) = e^{ix\beta}\Phi(\beta) - e^{ix\alpha}\Phi(\alpha) - ix\int_\alpha^\beta e^{ixu}\Phi(u)du.$$

If there exists a sequence $\{a_n\}$, $a_n \uparrow \infty$, *such that for all* $b>0$ *and for all real* x,

$$(4)\qquad \lim_{n\to\infty}\int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} e^{ixu}d\Phi(u),$$

then $\Phi(x)$ *is called a Fourier transform of order* 1 *of* $f(x)$.†

Definition 2 looks very much the same as definitions used by Hahn, Wiener, and Bochner.‡ There is one essential difference, however, due to the fact that Definition 2 does not a priori specialize the behavior of $f(x)$ at infinity, while other definitions do so. From the example at the end of this note it will be seen that there exist cases in which Definition 2 can be applied while no Bochner transform of any order can be defined.

Definition 2 is justified not only by the fact that it generalizes Definition 1 to which it reduces when $\Phi(x)$ is absolutely continuous, but also by the following theorem.

THEOREM 2. *If* $f(x)$ *is integrable over every finite range and if*

$$\lim_{a\to\infty}\frac{1}{\pi}\int_{-a}^{a} f(t)\,\frac{\sin b(x-t)}{x-t}\,dt$$

exists for each x, *then the Fourier transform of order* 1 *of* $f(x)$ *is uniquely determined up to an additive constant whenever it exists at all.*

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† This definition closely resembles the one we used in our note, *On a theorem of Paley and Wiener*, Annals of Mathematics, (2), vol. 34 (1933), pp. 606–614, especially pp. 609–611. We intend to discuss the relationship between these two definitions elsewhere. A definition identical to that of our Annals paper was successfully used in a recent paper by Verblunsky, *Trigonometric integrals and harmonic functions*, Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 1–48.

‡ See Wiener, *Generalized harmonic analysis*, Acta Mathematica, vol. 55 (1930), pp. 117–258, especially p. 159; Bochner, *Vorlesungen über Fouriersche Integrale*, 1932, Chapter 6: other references are also given there.

In particular, if $f(x) \subset L_p$, $1 \leqq p < \infty$, *then the Fourier transform of order* 1 *of* $f(x)$ *is given by*

$$(5) \qquad \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t) dt, \qquad (x \neq 0).\dagger$$

Furthermore, if $p > 1$, *the function*

$$(6) \qquad F(x, b) = (2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u)$$

converges in L_p *to* $f(x)$ *as* $b \to \infty$.

Applying the same argument as in the proof of Theorem 1, we now have

$$(7) \quad (2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u) = \lim_{a \to \infty} \frac{1}{\pi} \int_{-a}^{a} f(t) \frac{\sin b(x - t)}{x - t} dt.$$

Hence if there exists another Fourier transform of order 1 of $f(x)$, say $\Phi_1(x)$, we must have, for all $b > 0$ and all x,

$$\begin{aligned} 0 &= \int_{-b}^{b} e^{ixu} d\Psi(u) \\ &= \Psi(b) e^{ixb} - \Psi(-b) e^{-ixb} - ix \int_{-b}^{b} e^{ixu} \Psi(u) du, \end{aligned}$$

where $\Psi = \Phi - \Phi_1$. For $x = 0$ this gives $\Psi(b) = \Psi(-b)$. Upon setting $\Psi_0(u) = \Psi(u) - \Psi(b)$, we now have

$$\int_{-b}^{b} e^{ixu} \Psi_0(u) du = 0.$$

Hence $\Psi_0(u) = 0$, $\Psi(u) = \Psi(b)$ almost everywhere in $(0, b)$. Since b is arbitrary we conclude that $\Psi(x) =$ const. for all $x \neq 0$.

Now assume that $f(x) \subset L_p$, $1 \leqq p < \infty$. Then in (7) we may replace $\lim_{a \to \infty} \int_{-a}^{a}$ by $\int_{-\infty}^{\infty}$, the integral being absolutely convergent.

The fact that $F(x, b) \to f(x)$ in L_p as $b \to \infty$, if $p > 1$, follows then from our lemma. Hence it remains only to prove that the

† The value of $\Phi(x)$ at $x = 0$ is irrelevant since it can be modified arbitrarily without changing the value of $F(x, b)$.

Fourier transform of order 1 of $f(x)$ exists and is given by (5). To do so we evaluate $F(x, b)$ with

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t)dt,$$

which is a continuous function of x. Thus we get

$$\begin{aligned} F(x, b) &= (2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u) \\ &= (2\pi)^{-1/2} \left[e^{ixu}\Phi(u)\right]_{u=-b}^{u=b} - (2\pi)^{-1/2} ix \int_{-b}^{b} e^{ixu}\Phi(u)du \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} \left[(e^{ib(x-t)} - e^{-ib(x-t)} - e^{ibx} + \frac{e^{-ibx}}{-it} f(t)\right]dt \\ &\quad - \frac{ix}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{-it} dt \int_{-b}^{b} (e^{iu(x-t)} - e^{ixu})du \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin b(x-t)}{x-t} dt = \lim_{a\to\infty} (2\pi)^{-1/2} \int_{-b}^{b} f(u, a)e^{ixu}du,\dagger \end{aligned}$$

which is precisely relation (4) of Definition 2.

Theorem 2 shows that a function $f(x) \subset L_p$, $1 \leqq p < \infty$, always has a Fourier transform of order 1, which is a continuous function of x. A natural question arises whether every $f(x) \subset L_p$ has a Fourier transform (in the sense of Definition 1). We shall prove that in general this is not the case, if $p > 2$.

THEOREM 3. *A necessary and sufficient condition that a function $f(x)$ should have a Fourier transform in the sense of Definition* 1, *is that the continuous function*

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t)dt \tag{8}$$

be absolutely continuous.

This result is easily derived by combining Definition 1 and Theorems 1 and 2. The question of the existence of a Fourier transform of $f(x)$ is thus reduced to the discussion of the abso-

† See an analogous computation in Wiener, loc. cit.

lute continuity of $\Phi(x)$ as given by (8). It is well known that if $1<p\leqq 2$, every $f(x)\subset L_p$ has a Fourier transform in L'_p, $p'=p/(p-1)$. This Fourier transform must coincide with the Fourier transform of Definition 1. Hence $\Phi(x)$ is absolutely continuous when $1<p\leqq 2$. This result can also be verified directly.

Now assume $p>2$. Consider L_p as a metric vector space with the usual definition of its metric. Referring to a previous result of ours† we see that the set of functions of L_p for which $\Phi(x)$ is of bounded variation in any interval $(0, A)$, $A>0$, is of the first category in L_p. By modifying slightly the argument there used we might even replace the interval $(0, A)$ by an arbitrary interval, no matter how small. Consequently we may state the following result.

THEOREM 4. *If $2<p<\infty$, the set of functions of L_p which possess a Fourier transform in the sense of Definition* 1 *is of the first category in L_p. If $1\leqq p\leqq 2$ every $f(x)\subset L_p$ has a Fourier transform, indeed a Fourier transform in $L_{p'}$.*

It is of interest to exhibit a sufficient criterion for the existence of the Fourier transform of $f(x)\subset L_p$, $p>2$, which is more general than that of your Theorem C which requires the boundedness in L_q of the set $f(x, a)$, $a\geqq 0$.

THEOREM 5. *Let $f(x)\subset L_p$, $1\leqq p<\infty$, and let, for all $a\geqq a_0\geqq 0$ and for almost all x,*

$$|f(x, a)| \leqq P(x), \tag{9}$$

where $P(x)$ is a positive measurable function, integrable over every finite range. Then $f(x)$ has a unique Fourier transform in the sense of Definition 1.

In view of Theorem 3 it suffices to establish the absolute continuity of $\Phi(x)$. Now

$$\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{-itx}-1}{-it}f(t)dt = \lim_{a\to\infty}\int_0^x f(t, a)dt,$$

and (9) shows that the set of functions $\int_0^x(t, a)dt$ is absolutely

† E. Hille and J. D. Tamarkin, *On moment functions*, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 902–908, especially p. 905.

continuous over every fixed finite range of x, uniformly in a. Hence the limit function $\Phi(x)$ is also absolutely continuous.

A striking example of an application of Theorem 5 is given by the following function†

$$f(x) = \begin{cases} x^{-\alpha} e^{ix \log x}, & x \geqq 2, \\ 0, & x < 2, \end{cases} \quad (0 < \alpha \leqq 1/2). \tag{10}$$

Here $f(x, a)$ converges to a continuous function $F(x)$, uniformly over every finite interval. However, for large positive x,

$$F(x) = \exp\left\{ i(\pi/4 - e^{x-1}) + (1/2 - \alpha)(x - 1)\right\}(1 + o(1)); \tag{11}$$

hence $F(x)$ does not belong to any Lebesgue class over $(-\infty, \infty)$ Nevertheless the condition of Theorem 5 is satisfied and $F(x)$ is the Fourier transform of $f(x)$ in the sense of Definition 1, while $f(-x)$ is the Fourier transform of $F(x)$ in L_p for $p > 1/\alpha$. This is not the worst possible behavior that $F(x)$ may exhibit under such circumstances. We have merely to replace the term $\log x$ in (10) by repeated logarithms of sufficiently high order to introduce iterated exponentials of arbitrarily high order in (11). In all cases $F(x)$ has a Fourier transform in L_p, hence also in the sense of Definitions 1 and 2. However, the integral $\int_{-\infty}^{\infty} |F(x)|/(1+|x|^k)dx$ does not exist, no matter what $k > 0$ is. Hence the Wiener-Bochner theory can not be applied to $F(x)$.

BROWN UNIVERSITY AND YALE UNIVERSITY

† E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, III, Mathematische Annalen, vol. 108 (1933), pp. 525–577, especially p. 575. The factor $(2\pi)^{1/2}$ should be crossed out in formula (18.22).