TWO BOOKS ON TOPOLOGY

Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie. By Ernst Steinitz; edited and completed by Hans Rademacher. Berlin, Springer, 1934. viii+351 pp.

Lehrbuch der Topologie. By H. Seifert and W. Threlfall. Leipzig and Berlin, Teubner, 1934. vii+353 pp.

These two books present an interesting contrast; both have to do with topology in a pure, or combinatorial form, but otherwise they have little in common. The one is a memoir; the other is a textbook. The one is an isolated chapter of geometry which bears little relation to the main stream of contemporary topological research, but which stands by itself, firm in its own intrinsic worth. It deals with a question almost as old as analysis situs itself—the combinatorial classification of ordinary polyhedra. The other is an exposition of the rudiments of modern topology—the homology theory of n-dimensional complexes and manifolds. It aims to give the student a glimpse of the far-reaching developments of present-day topology, and to acquaint him with the vital ideas at their root.

The book by Steinitz is a posthumous memoir, appearing as the forty-first volume of the familiar yellow-covered series put out by Springer. It carries on the study of ordinary (two-dimensional) polyhedra already begun by Steinitz in his *Enzyklopädie* articles on *Polyeder und Raumeinteilungen.* The theory develops about a single central problem: Are there necessary and sufficient combinatorial conditions in order that it be possible to realize geometrically a two-dimensional complex by a convex polyhedron? In other words, can convex polyhedra be characterized in a purely combinatorial fashion? This problem has a classical ring; it is a natural outgrowth of the pioneer work of Euler on convex polyhedra; yet here it receives for the first time a complete and definite affirmative answer. The book is self-contained, well-illustrated, and easy to read. One cannot go far in it without sensing the enthusiasm of an author who has made his subject a hobby as well as a serious piece of mathematics. Steinitz' fame may rest on his contributions to algebraic field-theory, but he evidently took just as much delight, if not more, in working with polyhedra.

The book is divided into three parts. The first part makes an exploration, partly historical and partly intuitive, of possible forms and types of polyhedra. It discusses the following topics: the Euler formula ($v-e+f=2$) and its extensions, polyhedral volume, the topological forms of surfaces (including the non-orientable), the Cauchy theorem on the rigidity of a convex polyhedron, and the Legendre determination of the number of degrees of freedom involved in the construction of a polyhedron of given type. At the end of this preliminary survey the problem of the combinatorial characterization of convex polyhedra is broached through polyhedra with triangular faces and the dual polyhedra with three-edged vertices.

The second part of the book presents a purely combinatorial theory of polyhedra, guided by the preceding exploration, but founded on strict abstract principles. A (two-dimensional) *complex* is defined as an abstract collection of "vertices," "edges," and "faces," bound together by specified "incidences." Selective restrictions are gradually imposed until complexes evolve which are regular enough to represent surfaces, then polyhedra, and finally convex polyhedra. The combinatorial counterpart of a closed surface, or a surface bounded by a number of non-intersecting curves, is a *normal complex* (known to topologists as a manifold). If two normal complexes have the same Euler characteristic and the same number of boundaries, and if both are orientable or non-orientable, it is possible to pass from the one to the other by a sequence of moves which split a face or edge in two or which perform an inverse operation, thus adding or subtracting an edge and either a face or a vertex. This is a combinatorial form of the well known classification of (two-dimensional) manifolds. A complex is called a *polyhedron* if it is normal except for singular vertices, and if it does not have more than one edge jointly incident with two faces or with two vertices. Such a polyhedron is said to be without *encroaching elements* if it possesses a pair of faces with two vertices in common only if the vertices are the end-points of a common edge. By appropriate composition of the splitting moves mentioned above it is possible to get "regular" moves which leave invariant the notions of *polyhedron* and *polyhedron without encroaching elements*. The combinatorial theory culminates in the *K-polyhedron*, which is both an Euler complex (that is, a normal complex with characteristic two) and a polyhedron without encroaching elements. A *K-polyhedron* possesses the important characteristic that it can be derived from a tetrahedron by regular splitting.

The third part of the book gives proofs, which are rigorously based on the foundations of geometry, that each K-polyhedron can be realized by a concrete convex polyhedron. Two of the proofs depend on the fact that a K-polyhedron can be derived from a tetrahedron by regular splitting. The first proof uses analytical considerations of linear independence to establish this, while the second uses purely geometrical considerations of incidence and order (without continuity). The third proof is projective; it also uses considerations of incidence and order (without continuity), but it depends on certain interesting new types of moves.

We are indebted to Rademacher for the complete and polished form in which this book comes to us. Steinitz had not finished the manuscript when he died in 1928. Rademacher had to fill a gap in the second part of the memoir and to write the last half of the third part.

The textbook of Seifert and Threlfall should do much to smooth the path of the student who wants to learn the fundamentals of (combinatorial) topology. The authors have concentrated on basic concepts and methods, avoiding generalizations; they have explained these concepts and methods in as simple and concrete a fashion as possible, yet one which is thorough and rigorous. The exposition proceeds by easy stages, with examples and illustrations at every turn. It presupposes nothing; even the necessary group theory is developed in a special supplementary chapter.

The book follows the lines laid down in Veblen's *Analysis Situs* (1922, 2d ed. 1931), van Kampen's dissertation (1929), and the early chapters of Lefschetz' *Topology* (1930). The influence of American topologists has been so strong that terminology hitherto confined to English has been translated; for example, "chains" finally appear as "Ketten." The influence of the German algebraic school is reflected in a frank use of group theory. A particularly valuable feature of the book is the attention paid to the fundamental (Poincaré) group, covering spaces, and three-dimensional manifolds; in no other single place in the literature has so much interesting information been gathered together on these topics. The investigations of the authors themselves find place in the chapter on three-dimensional manifolds.

The first two chapters are largely preparatory; the first draws on intuition to create a suitable topological atmosphere, the second defines the geometrical foundations on which the subsequent work is built. The idea of neighborhood is introduced to permit the definition of a continuous mapping of one figure on another (any neighborhood on the second figure contains the map of a neighborhood on the first). Then simplexes appear, and complexes are defined as "neighborhood-spaces" which can be cut into simplexes.

Chapters 3–5 deal with the fundamental topological invariants of an n-complex: Betti numbers, torsion coefficients, dimension. In developing the homology theory the authors give simplicial chains precedence over singular ones; they turn to the latter only when it is necessary to establish the topological invariance of the homology groups of a complex. This is the customary procedure, but the reviewer feels that if it were reversed the theory would gain a better balance and readier comprehension. The notion of singular chain, if taken at the beginning, would no longer seem at all artificial, and simplicial chains would follow naturally as a special case. Instead of the usual invariance theorem one would have the important fact that the homology groups of a complex can be reduced to groups with finite simplicial bases.

The authors confine their attention to chains and cycles with integral coefficients or coefficients mod 2, but they give plenty of illustrative material; homology groups are worked out in full for several sample complexes. Despite the criticism above it must be admitted that the proof of the topological invariance of the homology groups of a complex is well done; appropriate emphasis is placed on the part played by deformations and simplicial approximation of continuous mappings. Further invariance proofs use local homology groups to establish the permanence of dimension, and other properties of a complex. The local groups appear again in an essential fashion when manifolds are defined.

The last six chapters of the book (excluding the supplementary chapter on group theory) treat topics connected with manifolds—first two-dimensional, then three-dimensional, and finally n-dimensional. The two-dimensional manifolds are classified by the methods of Brahana (although the text does not attribute them to him). Then, after a digression on the fundamental (Poincaré) group of a complex and a further digression on covering complexes, there is a chapter on methods of constructing and analyzing three-dimensional manifolds.

A (closed) n-dimensional *manifold* is defined to be a connected n-complex at each point of which the local homology groups are those of an $(n-1)$-sphere; this definition expresses the essential combinatorial character of a manifold in a topologically invariant manner. The Poincaré duality theorem is established, as well as the standard theorems on intersection and linking numbers. A final chapter deals with the Brouwer degree of a mapping, and the Lefschetz fixed point formula treated by the methods of Hopf.

The book contains a very complete bibliography, it is well indexed, and as a further help to the reader most chapters are prefaced by a summary of their contents. At many points in the text the reader is referred to supplementary notes collected at the end of the book, which indicate extensions of the theory and links with other work. Little attempt is made, outside these notes, to attribute ideas to their originators; this is particularly glaring when the ideas are heavily exploited. The authors apologize for their failure to treat the Alexander duality theorem and the theory of compact metric spaces (so beautifully rounded out by the recent work of Pontrjagin), but they promise to write a second volume on these matters if someone else does not do so first.

A. W. Tucker

WEDDERBURN ON MATRICES

Lectures on Matrices. By J. H. M. Wedderburn. American Mathematical Society Colloquium Publications, volume 17. New York, American Mathematical Society, 1934. vii+200 pp.

For the past seventy-five years matric theory has been growing in stature and in favor among men. Many branches of mathematics have been more promising infants, but few have shown such sustained growth and ever widening field of application. The concept of matrix, like that of group, extends its roots under algebra, number theory, geometry, differential and integral equations, wave mechanics—a fairly representative cross-section of modern mathematics. This fundamental nature of matric theory has never been so generally appreciated by mathematicians as at present. Thus Wedderburn's book is timely.

An evident fact in the history of matric theory is that the important theorems are not due to any small group of men. A few names stand out prominently, of course, but it has taken close to a thousand distinct contributions to bring the theory to its present state. Why this was so is not evident, but it must be true that the theorems which now seem so clear to us were not intuitive to mathematicians at the time of their discovery. It has been the common history of the important theorems that they were discovered first in special cases, then generalized and laboriously proved, and finally furnished with neat direct proofs. In the book under review most of the theorems have reached the last stage of development, and the reader is apt to be unaware of the amount of publication which it renders obsolete.

In recent years Wedderburn has been one of the most important contributors to matric theory. His discoveries have been published as they came, and have taken their place in matric lore. It would be absurd, therefore, to expect a large proportion of the results in this book to be new. But theorems have been extended, sharpened, and clarified to a remarkable degree.

It is the organization and presentation of the material, however, which