ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

264. Professor C. G. Latimer: *On the fundamental number of a rational generalized quaternion algebra.*

Let $\mathfrak{A}$ be a rational generalized quaternion algebra with a basis 1, i, j, ij, where $i^2=-\alpha$, $j^2=-\beta$, $ij=-ji$; α and β being integers, neither divisible by the square of a prime. Brandt defined the fundamental number d of $\mathfrak{A}$ and showed that two algebras with the same d were equivalent. (*Idealtheorie in Quaternionenalgebren*, Mathematische Annalen, vol. 99 (1928), pp. 9–12.) In this paper, d is determined explicitly in terms of certain divisors of α and β. In a recent paper, Albert showed that $\mathfrak{A}$ has a basis in the above form with $\alpha=-\tau$, $\beta=-\sigma$ where τ, σ satisfy certain conditions. (*Integral domains in rational generalized quaternion algebras*, this Bulletin, vol. 40 (1934), pp. 167–168.) As an application of the present theorem, it is found that Albert's $\sigma=-d$ and that his τ may be any integer satisfying the conditions mentioned above. (Received May 15, 1935.)

265. Professor E. T. Bell: *General relations between Bernoulli, Euler, and allied polynomials.*

The relations in question form a complete set, connecting any $n(n=1, 2, 3, 4)$ of the polynomials, which appear in the relations as the arguments of arbitrary functions, the arguments being linear in the polynomials (precisely, in the umbrae of the polynomials). There are also general relations connecting polynomials whose ranks are in arithmetical progression. By setting the variable equal to zero or one in the relations, corresponding relations are obtained for the associated numbers. (Received May 18, 1935.)

266. Mr. Garrett Birkhoff: *Abstract continuous groups.* I: *Linear spaces.*

It is shown that functions between any linear spaces occurring in practice can be analyzed by the methods of the calculus. The most interesting new developments are, the description of these spaces by direct *combination* of norm functions, a related definition of the "order" of a function, a characterization of polynomials by functional equations. (Received May 18, 1935.)

267. Professor R. L. Wilder: *The strong symmetrical cut-sets of closed euclidean n-space.*

It was recently shown by W. Dancer (this Bulletin, vol. 41 (1935), p. 342), that the simple closed curve is the only strong symmetrical cut-set of the simple closed surface. It is the purpose of the present note to extend this result as follows: Let M be a strong symmetrical cut-set of the closed euclidean n-space. Then M is a generalized closed $(n-1)$-manifold (Annals of Mathematics, vol. 35 (1934), p. 878), whose Betti numbers $p^i(M)$, $0 \leqq i \leqq n-2$ are all zero. In particular, if $n=3$, M is the topological 2-sphere. (Received May 21, 1935.)

268. Professor E. P. Lane: *The neighborhood of a sextactic point on a plane curve.*

In this paper it is shown that, by suitable choice of the projective coordinate system, the equation representing an analytic plane curve in the neighborhood of a sextactic point (0, 0) on it, can be reduced to the form $y=x^2+a(x^6+x^8)+a_9x^9+a_{10}x^{10}+\cdots$, $(a\neq 0)$. This canonical expansion is then used to study the curve in the neighborhood of the sextactic point, particular attention being given to certain cubic, quartic, and quintic curves having contacts of various orders with the curve at the point. (Received May 25, 1935.)

269. Professor G. A. Baker: *The probability that the mean of a second sample will differ from the mean of a first sample by less than a certain multiple of the standard deviation of the first sample.*

It is shown that the variable $v=(x-z)/y$ where x is the mean of a first sample of n_1, z is the mean of a second sample of n_2, and y is the standard deviation of the first sample is distributed as proportional to $r^{(1-n_1)/2}$ where $r=1+[n_2v^2/(n_1+n_2)]$. The sampled population is assumed to be normal. If the two samples are combined to obtain a value of y then the corresponding distribution is $s^{-(n_1+n_2)/2}$ where $s=1+[n_1n_2v^2/(n_1+n_2)^2]$. These distributions are easily transformed into "Student's" t-distribution. They can be used to calculate "true" probable errors, to test the significance of divergence of samples, and as rejection criteria. (Received May 29, 1935.)

270. Professor G. A. Baker: *The probability that the standard deviation of a second sample will differ from the standard deviation of a first sample by less than a certain multiple of the standard deviation of the first sample.*

The distribution function of $v=(x-y)/x$ where x is the standard deviation of a first sample of n_1 and y is the standard deviation of a second sample of n_2 is shown to be proportional to $(1-v)^{(n_2-2)}\{[(1-v)^2n_2/n_1]+1\}^{(2-n_1-n_2)/2}$ if the sampled population is normal. This distribution allows one to calculate "true" probable errors which are greatly in excess of conventional probable errors of standard deviations for small samples. (Received May 29, 1935.)

271. Dr. L. S. Bosanquet: *Some arithmetic means connected with Fourier series.*

In the theory of the Cesàro summability of a series Σa_n an important rôle is played by the sequence na_n. This may be traced, to a great extent, to the known identities $\tau_n^\alpha = \alpha(s_n^{\alpha-1} - s_n^\alpha) = n(s_n^\alpha - s_{n-1}^\alpha)$, where s_n^α and τ_n^α are the Cesàro means of order α of the sequences $s_n = a_0 + a_1 + \cdots + a_n$ and na_n, respectively. The object of this paper is to investigate the Cesàro means of the sequence na_n, where $\Sigma a_n \cos nt$ is a Fourier series. In particular, necessary and sufficient conditions are obtained for the function to satisfy in order that the sequence na_n should be bounded (C). (Received June 3, 1935.)

272. Dr. L. M. Blumenthal (National Research Fellow): *The metric characterization of the n-dimensional hyperbolic space.*

In this paper the n-dimensional hyperbolic space $H_{n,r}$ of curvature $-1/r^2$ is characterized among semimetric spaces by means of relations between the distances of its points. If $p_1, p_2, \cdots, p_k$ are k points of a semimetric space, and we call the symmetric determinant $|\cosh p_ip_j/r|$, $(i, j=1, 2, \cdots, k)$, the Lobatschefskian determinant of the k points, it is proved that a semimetric space S is congruent with a subset of $H_{n,r}$ if and only if (i) for every integer $s \leqq n+1$, the Lobatschefskian determinant of each set of s points of S has the sign $(-1)^{s-1}$ or vanishes; (ii) the Lobatschefskian determinant of each set of $n+2$ points of S vanishes; while, in case S consists of exactly $n+3$ points, we adjoin (iii) the Lobatschefskian determinant of the $n+3$ points vanishes. If we suppose that the space S is convex, complete, and of finite dimension, conditions that it be isometric with a convex subset of $H_{n,r}$, may be given in terms of Lobatschefskian determinants formed for merely each quadruple of its points, after the manner of the author's *Concerning spherical spaces* (American Journal of Mathematics, vol. 57 (1935), pp. 51–61). (Received June 4, 1935.)

273. Professor V. G. Grove: *Differential geometry of a certain type of surface in S_4.*

The purpose of this paper is to study surfaces sustaining an orthogonal conjugate net and immersed in a space of four dimensions. As is well known the normals to the surface lie in a unique normal plane. The author studies the given surface by the use of certain surfaces in three dimensions called *normal projection surfaces.* Each normal to the surface determines a unique normal projection surface. Among the normals to the given surface there are two unique normals which are perpendicular and are such that the normal projection surfaces determined by them have the one a maximum and the other a minimum total curvature. These unique principal normals permit writing the system of partial differential equations in a canonical form. Certain analogues of theorems concerning surfaces in three dimensions are given. For example, a curve is a geodesic if and only if the osculating plane of the curve at each of its points intersects the normal plane of the surface at these points in a line. If this line generates a congruence conjugate to the congruence, the geodesic is a plane curve. (Received June 7, 1935.)

274. Miss M. C. Wolf: *Symmetric functions of matrices.*

This paper modifies the fundamental theorem concerning elementary symmetric functions so that it may be applied to matrices. Since the commutative law of multiplication does not hold in general, it has been necessary to define sets of fundamental symmetric polynomials which are a direct generalization of the elementary symmetric functions, containing them as a special case when the variables are commutative. Any polynomial of degree m symmetric in the non-commutative elements $x_1, x_2, \cdots, x_n$ is equal to a polynomial, with integral coefficients, in the elements of any fundamental set and the coefficients of the original polynomial. The representation is unique. In the commutative case n elementary symmetric functions are sufficient to represent the symmetric polynomials of any degree m involving n variables. In the general case this does not hold; an additional finite set of fundamental polynomials is necessary for each higher degree. Although the choice of a fundamental set is not unique, the number for each degree is unique. A method for calculating these numbers is given. For the first five degrees they are 1, 1, 2, 6, 22. One particular simple fundamental set is defined and discussed. (Received June 8, 1935.)

275. Mr. H. J. Hamilton: *Transformations of multiple sequences.*

The double sequence $\{\sigma_{mn}\}$, where $\sigma_{mn} \equiv \sum_{k,l=1}^{\infty} a_{mnkl}s_{kl}$, is said to be the transform of the double sequence $\{s_{kl}\}$ by means of the infinite matrix $\|a_{mnkl}\|$. Definitions of ultimate boundedness, boundedness, convergence, bounded convergence, ultimately regular convergence (that is, convergence plus ultimate row and column convergence), bounded ultimately regular convergence, and regular convergence being given, it is possible to find necessary conditions on the matrix $\|a_{mnkl}\|$ in order that $\{\sigma_{mn}\}$ have a specified one of the above properties whenever $\{s_{kl}\}$ has a specified one of these properties, with existence of the transform for each m, n hypothesized; and to find sufficient conditions for the same situations, with or without existence of the transform for each m, n. In this paper such conditions are determined for the analogous transformations of n-tuple sequences in general. Several auxiliary types of sequences are introduced, and there is given a discussion of limit-preservation when $\{s_{kl}\}$ is convergent. (Received June 8, 1935.)

276. Professor E. T. Browne: *On the matric equations* $P(A, X) = 0$ and $P(X) = A$.

Let A be a given square matrix of order n, and let $F_i(A)$, $(i=0, \cdots, m)$, be given polynomials in A with scalar coefficients. The purpose of this paper is to investigate the existence of square matrices X of order n satisfying the equation $P(A, X) \equiv \sum_{i=0}^{m} F_i(A)X^{m-i} = 0$. In the particular case where $F_i(A)$, $(i=0, \cdots, m-1)$, reduce to constants p_i, and $F_m(A) = p_m - A_m$ the above equation reduces to the special case $P(X) = A$. In this paper only such matrices X as are expressible as polynomials in A are considered, and it is shown that the principal *idempotent* and *nilpotent* matrices associated with A lend themselves very readily to a simple and elegant solution of the problem. (Received June 11, 1935.)

277. Professor Leonard Carlitz: *On the representation of a polynomial in a Galois field as the sum of an odd number of squares.*

The problem is that of determining the number of sets of polynomials X_i, $(i=1, \cdots, 2s+1)$, each of degree k such that $\alpha L=\alpha_1 X_1^2+\cdots+\alpha_{2s+1}X_{2s+1}^2$, where $\alpha, \alpha_1, \cdots, \alpha_{2s+1}$ are in the Galois field, and $\alpha_1+\cdots+\alpha_{2s+1}=0$ or $\neq 0$ according as $l=$ degree of $L<$ or $=2k$. In the present paper the results of an earlier communication are extended in various directions. The α_i are now quite arbitrary. In the case $l<k$, $L=L_0M^2$, where L_0 contains no quadratic factors, it is shown that the ratio of the number of representations for L to the number for L_0 is a divisor function of M. (Received June 17, 1935.)

278. Professor Leonard Carlitz: *On sums of squares of polynomials.*

This note supplements two papers on the representation of a polynomial in a Galois field as the sum of an assigned number of squares. The problem considered here is that of finding the number of solutions of $\alpha_1 X_1^2+\cdots+\alpha_s X_s^2=0$ in polynomials X_i of degree k; the α_i are in the Galois field. (Received June 17, 1935.)

279. Professor Leonard Carlitz: *On certain higher congruences.*

This paper deals with the congruence $\pi(t-G)\equiv A \pmod{P}$: A, P, G, denote polynomials in an indeterminate x with coefficients in a fixed Galois field. The product extends over all G (zero included) of degree less than some fixed m. The congruence has either no solutions at all, or else has p^{nm} distinct solutions, where p^n is the order of the Galois field, m is less than the degree of P, and P is assumed to be irreducible. Simple criteria for the solvability of the congruence are given. (Received June 17, 1935.)

280. Dr. Deane Montgomery (National Research Fellow) and Dr. Leo Zippin: *Periodic one-parameter groups in three-space.*

In this paper a one-parameter group of homeomorphisms of euclidean three-space E into itself is considered. A certain weak periodic character with respect to the points of E is postulated for the group, and it is shown to be closely related to the rotation group of three-space, the fixed points constituting an infinite topological line which acts as the axis of a quasi-rotation. The present theorem may be formulated as follows: A continuous, one-parameter group $T(x; t)$, x in E, t any real number, of point and wise-periodic homeomorphisms of E into itself, whose period-function is bounded in every sphere, is a topological quasi-rotation group. In particular, it is the topological rotation group whenever the period-function is constant over the moving points. (Received June 13, 1935.)

281. Professors H. E. Buchanan and W. L. Duren: *On the characteristic exponents in certain types of problems of mechanics*

H. E. Buchanan published in the American Journal, vol. 45 (1923), and vol. 50 (1928), a discussion of the stability of the straight line and the equilateral triangle positions in the problem of three finite masses. He discussed in the American Mathematical Monthly, vol. 38 (1931) and vol. 40 (1933), the stability of the so-called helium atom for the straight line and equilateral triangle solutions. In all four of these problems the characteristic exponents 0, 0, $\pm i\omega$ occurred, ω being the angular velocity of rotation of the system. This paper gives the conditions under which these exponents will occur when the coordinates of the three bodies are referred to axes rotating uniformly about the Z-axis with angular velocity ω. This paper shows that if the equations of motion of the system have an integral which is such that the corresponding integral of the equations of variation is periodic in t with minimum period T, incommensurable with t in the case of generalized equilibrium, then two of the characteristic exponents are $\pm 2\pi i/T$. The differential equations of the three body problem and of the helium atom have ten integrals. The center of gravity integrals are used to eliminate six of the variables. Of the four integrals remaining, two are independent of the time and thus account for the exponents 0, 0. The other two integrals are periodic with period $2\pi/\omega$ and hence require that the exponents $\pm i\omega$ occur. (Received June 20, 1935.)

282. Professor R. P. Agnew: *Generalizations of the Riemann-Lebesgue theorem*. I.

The well known Riemann-Lebesgue theorem states that if $\phi(x)\equiv x$, $f(x)\mathfrak{E}L$, and $F(u)=\int_{-\infty}^{\infty}f(x)e^{iu\phi(x)}dx$, then $\lim_{u\to\infty}F(u)=0$. In this paper we show that the class of real measurable functions $\phi(x)$ having the property that for each $f\mathfrak{E}L$ and $\varepsilon>0$ there is a constant M such that $\int_B^{B+h}|F(u)|du<\varepsilon h+M$ for all B and all $h\geqq 0$ is identical with the class of dispersed functions $\phi(x)$, i.e. real measurable functions $\phi(x)$ having the property that for each constant c the set of values of x for which $\phi(x)=c$ has measure 0. We give related results, and an application to Fourier transforms of functions of bounded variation. (Received June 21, 1935.)

283. Dr. G. de B. Robinson: *On the fundamental region of an orthogonal representation of a finite group*.

If H is a sub-group of G, then G may be represented as a permutation group on the cosets of H. Such a representation gives rise to a set of permutation matrices which constitute a reducible representation of G, an irreducible component (α) appearing with a multiplicity $S_H{}^\alpha=(1/h)\Sigma\chi_R{}^\alpha$ where the summation extends over those R's contained in H. In the present paper it is shown that the functions $S_H{}^\alpha$ have an important geometrical significance, and yield information concerning the definiteness of the fundamental region of the corresponding irreducible representation, if we suppose it to be orthogonal. These ideas lead naturally to a geometrical interpretation of the characteristic, which has played no part in the geometrical theory as originated by Klein, though it is fundamental from the algebraic point of view. (Received June 24, 1935.)

284. Dr. H. S. Grant: *A generalization of a cyclotomic formula.*

Jacobi stated without proof (Journal für Mathematik, vol. 30 (1846), p. 167) the following cyclotomic formula: $F(-1)\ F(\alpha^2)=\alpha^{2m}\ F(\alpha)\ F(-\alpha)$, where $F(\alpha)=x+\alpha x^g+\alpha^2x^{g^2}+\cdots+\alpha^{q.-2}x^{g^{q.-2}}$, $q.$ is an odd prime, g is a primitive root, mod $q.$, $g^m\equiv 2$, mod $q.$, $x^{q.}=1$ $(x\neq 1)$, $\alpha^{q.-1}=1$ $(\alpha\neq 1)$. This relation reduces essentially to one connecting two Jacobi ψ-functions. By making use of the generalized Jacobi-Kummer cyclotomic function, and its prime ideal factorization (H. H. Mitchell, Transactions of this Society, vol. 17 (1916) pp. 165–177), this formula is extended to the case $q.^t\equiv 1$, mod n, $q.$ an odd prime, and t any exponent for which the congruence holds, n even. In the above formula $t=1$, $n=q.-1$. (Received June 28, 1935.)

285. Professor A. A. Albert: *Simple algebras of degree p^e over a centrum of characteristic p.*

The author considers normal simple algebras A of degree p^e over F of characteristic p. It is assumed that F is such that not every such A is a total matric algebra, and it is proved that A is cyclic if and only if A has a maximal sub-field $F(y)$, $y^{p^e}=\nu$ in F. As a consequence, a direct product of two such cyclic algebras is always cyclic. Moreover every normal simple algebra A of degree p^e over F of characteristic p is similar to a cyclic algebra (of degree p^q). The exponent of A is proved to be p^t, where t is the least exponent of all splitting fields $F(y_1, \cdots, y_r)$, $y_ip^t=d_i$ in F, of A. Finally A has exponent p^t if and only if A is similar to a direct product of cyclic normal division algebras D_i of degrees $p^{e_i}\leqq p^t$ over F, such that D_1 has degree p^t, the exponent of each D_i being its degree. (Received July 1, 1935.)

286. Dr. S. C. Kleene: *General recursive functions of natural numbers.*

Using a definition of general recursive function due to Herbrand and Gödel, the author shows that every general recursive function is expressible in the form $\psi\{\epsilon y[\rho(x_1, \cdots, x_n, y)=0]\}$ where ψ and ρ are ordinary recursive functions (that is defined from 0 and $x+1$ by substitutions and recursions on one variable) and $(x_1, \cdots, x_n)\ (Ey)\ [\rho(x_1, \cdots, x_n, y)=0]$; and conversely. The systems of equations which define recursive functions under this general definition cannot be recursively enumerated, since a recursive enumeration of them would make possible the construction of a new recursive function by diagonalizing. Likewise, no recursive process of deciding which systems define recursive functions is attainable. Since the condition that a system define a recursive function of one variable is expressible in the form $(x)\ (Ey)\ [\rho(s, y)=0]$, this gives a somewhat different proof than Gödel's that there are undecidable propositions in any mathematical logic satisfying certain conditions. For otherwise the logic could be used to decide recursively which systems of equations define recursive functions of one variable. Every problem of the form, whether $(x)\ (Ey)[\sigma(x, y)=0]$ holds $[\sigma(x, y)$ recursive$]$, is included in the general problem, which systems of equations define recursive functions of one variable. There are also non-recursive functions of the form, $\tau(x)=0$ or 1 according as $(y)\ [\rho(x, y)=0]$ holds or does not hold. (Received July 1, 1935.)

287. Dr. S. C. Kleene: *λ-definability and recursiveness.*

Let a function of positive integers be called λ-definable if it is formally definable in the sense of the author (American Journal of Mathematics, vol. 57 (1935), p. 219). Using the general definition of recursive function due to Herbrand and Gödel (see abstract 41-7-286), the author shows that every recursive function is λ-definable, and conversely. The Gödel method of representing functions by numbers makes possible the extension of these results (under certain restrictions) to the case of functions of which the values are well-formed formulas. (Received July 1, 1935.)

288. Dr. S. S. Wilks: *The sampling theory of systems of variances, covariances, and intraclass covariances.*

By means of factoring and transforming characteristic functions of second order symmetric functions of normally distributed variables, a method is given for determining the sampling distributions of certain systems of variances, covariances, and intraclass covariances with special application to the generalized intraclass correlation problem for m "families" of k variates each. The method is applied to the problem of determining all of the independently distributed sums of squares associated with layouts of the Latin square, random block, and equalized random block types in replicated agricultural experiments. (Received July 2, 1935.)

289. Professor R. S. Burington: *On the direct sum in circuit theory.*

The concept of direct sum is introduced into the matric theory development of electric circuit theory to clarify the theory relating to the interconnection of a finite number of networks. The present treatment should meet certain objections to the use of the "connection tensor" appearing in the writings of Quade, Cauer, Kron, and others. (Received July 3, 1935.)

290. Professor A. F. Moursund: *On the Abel-Poisson summability of derived series of the conjugate Fourier series.*

This note gives theorems for the rth derived series of the conjugate Fourier series which are analogous to theorems given by B. N. Prasad (Journal de Mathématiques Pures et Appliquées, vol. 11 (1932), Theorem 3) and A. Plessner (Mitteilungen des Mathematischen Seminars der Universität Giessen, 10 (1923), Theorem 3) for the conjugate Fourier series. (Received July 5, 1935.)

291. Professor G. T. Whyburn: *Concerning rationality bases for curves.*

In a recent paper (Monatshefte für Mathematik and Physik, vol. 42 (1935), pp. 37-44) Knaster shows that in any stably regular curve (that is a curve which remains regular on the addition of any regular curve) every rationality basis is also a regularity basis; and he raises the question as to whether this property characterizes the stably regular curves among the rational curves. This question may be answered in the negative, because it is possible to prove that *every dendrite* has this property and it is well known that not all dendrites are stably regular. However, in the present paper it is shown further that if a rational curve H enjoys this property then every true cyclic element of H is stably regular. Thus, by extending Knaster's result somewhat we obtain the following characterization: In order that a rational curve H should have the property that every rationality basis in H be a regularity basis in H it is necessary and sufficient that H be locally connected and that every true cyclic element be stably regular. (Received July 7, 1935.)

ERRATUM

Volume 41, page 331, abstract no. 200 (by Professor C. N. Moore): in the next to the last sentence, "$(N; c)$ to UV" should be replaced by "$(N; C)$ to UV, where $C_n = c_0 + c_1 + \cdots + c_n$."