SOME THEOREMS ON DOUBLE LIMITS*

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1. *Introduction.* Let $f(x, y)$ be an arbitrary single-valued real function of the real variables x, y defined in the neighborhood of a point $Q(a, b)$, which for simplicity may be taken as $(0, 0)$. The following sufficient (and obviously necessary) condition for the existence of the double limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x, y) \tag{1}$$

has been established.

THEOREM 1 (Clarkson).‡ *If $f(x, y)$ has a unique limit as $P(x, y)$ approaches Q on every curve having a tangent at Q, the double limit* (1) *exists.*

The present note is concerned with similar theorems, and for definiteness we state at the outset that the assertion, "$f(P)$ has a limit λ as $P\to Q$ on a point set§ E having Q as a limit point (or $\lim_{P\to Q} f(P)=\lambda$, on E)" shall mean that for each $\epsilon>0$ there exists a positive $\delta(\epsilon, E)$ such that $|f(P)-\lambda|<\epsilon$ for all points P of E satisfying the condition $0<|x|+|y|<\delta$.

Theorem 1 naturally suggests a question which is answered by Lemma 1, for convenience in the statement of which we introduce the following definition.

DEFINITION OF PROPERTY L. A class $\{E\}$ of sets E, each having Q as a limit point, will be said to have Property L if and only if any set S whatsoever of points having Q as a limit

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† I gratefully acknowledge my indebtedness to Mr. Hugh J. Hamilton for suggesting Lemma 1, and to Mr. Nelson Dunford for Theorem 5.

‡ Clarkson, *A sufficient condition for the existence of a double limit*, this Bulletin, vol. 38 (1932), pp. 391–392. A theorem essentially the same has been proved byVerčenko and Kolmogoroff, *Über Unstetigkeitspunkte von Funktionen zweier Veränderlichen*, Comptes Rendus, Académie des Sciences, URSS, new series, vol. 1 (1934), pp. 105–107.

§ In particular, on a curve.

point has a subset S^* which is contained in some one of the sets E and has Q as a limit point.

LEMMA 1. *A necessary and sufficient condition that the relation* $\lim_{P\to Q} f(P)=\lambda$ *on every set* E *of a class* $\{E\}$ *shall imply the existence of* (1) *is that* $\{E\}$ *have Property* L.

This lemma, whose proof we leave to the reader, provides a criterion for determining whether or not an analog of Theorem 1 holds for other classes of curves or point sets.

2. *The Class of Curves* $\{\mathfrak{A}\}$. Let $\phi(s)\equiv\sum_{n=1}^{\infty}a_n s^n$, $\psi(s)\equiv\sum_{n=1}^{\infty}b_n s^n$ be any two real power series with positive radii of convergence (say) ρ_a, ρ_b, respectively, and let ρ be chosen so that $0<\rho<\min(\rho_a,\rho_b)$. Then the equations

$$x=\phi(s),\qquad y=\psi(s),\qquad (|s|\leqq\rho), \tag{2}$$

define a curve $\mathfrak{A}$ through Q. We denote by $\{\mathfrak{A}\}$ the class of all such curves.

THEOREM 2. *The existence of a unique limit for* $f(P)$ *as* $P\to Q$ *on every curve of* $\{\mathfrak{A}\}$ *does not imply the existence of* (1).

PROOF. Let us assume the contrary, which implies that $\{\mathfrak{A}\}$ has Property L. We choose S as the set of points on the curve $y=e^{-1/x^2}$ for $x>0$, and proceed to show that the definition of Property L is not satisfied. Suppose that there exists a curve $\mathfrak{A}^*$ of $\{\mathfrak{A}\}$ and an infinite subset S^* of S of points $(\xi_n,\eta_n)\to(0,0)$, such that S^* lies on $\mathfrak{A}^*$. Then if (2) is the representation of $\mathfrak{A}^*$, there must exist at least one value of s, say σ_n, for which $\phi(\sigma_n)=\xi_n$, $\psi(\sigma_n)=\eta_n$, $(n=1,2,3,\cdots)$. Let λ be any limit point of the sequence $\{\sigma_n\}$, and let $\{s_n\}$ be a subsequence of $\{\sigma_n\}$ such that $s_n\to\lambda$ as $n\to\infty$. If $\{(x_n,y_n)\}$ is the corresponding subset of $\{(\xi_n,\eta_n)\}$, we have $0<x_n=\phi(s_n)\to 0$, and $0<y_n=\psi(s_n)\to 0$, whence by continuity $\phi(\lambda)=\psi(\lambda)=0$. Consequently, in view of the relation $|\lambda|\leqq\rho<\min(\rho_a,\rho_b)$, $\phi(s)$ and $\psi(s)$ have expansions of the form

$$\begin{aligned}\phi(s)&=\sum_{n=\mu}^{\infty}\alpha_n(s-\lambda)^n, &&(\mu\geqq 1,\ \alpha_\mu\neq 0),\\ \psi(s)&=\sum_{n=\nu}^{\infty}\beta_n(s-\lambda)^n, &&(\nu\geqq 1,\ \beta_\nu\neq 0),\end{aligned}\tag{3}$$

for $|s-\lambda|$ sufficiently small. Choose an integer m to satisfy the inequality $m\mu > \nu$, and consider the equation

$$\frac{\psi(s_n)}{[\phi(s_n)]^m} = \frac{e^{-1/x_n^2}}{x_n^m}, \qquad (n = 1, 2, 3, \cdots),$$

which is implied by $S^* \subset \mathfrak{A}^*$. Using (3) one sees that the left side increases without limit as $n \to \infty$, while the right side tends to zero. This contradiction completes the proof.

3. *The Class of Curves* $\{\mathfrak{B}_r\}$. Let r be a preassigned real number, or ∞, and denote by $\{\Gamma_r\}$ the class of all single-valued functions of $z(=s+it)$, each of which (i) is analytic in the extended plane except for a singularity at $z=r$, (ii) vanishes at $z=0$, and (iii) is real on the real axis. Then about $z=0$ each function in $\{\Gamma_r\}$ admits a power series expansion with real coefficients whose radius of convergence is $|r|$. Let $\{\Pi_r\}$ be the class of all such power series, and let $\{\mathfrak{B}_r\}$ be the class of all curves $\mathfrak{B}_r$ through Q each of which is defined parametrically by

$$(4) \qquad x = \phi(s) \equiv \sum_{n=1}^{\infty} a_n s^n, \qquad y = \psi(s) \equiv \sum_{n=1}^{\infty} b_n s^n,$$

where the power series belong to the class $\{\Pi_r\}$.

THEOREM 3. *For each fixed* r, $(0<|r| \leqq \infty)$, *the existence of a unique limit for* $f(P)$ *as* $P \to Q$ *on every curve of* $\{\mathfrak{B}_r\}$ *implies the existence of* (1).†

This theorem is an immediate consequence of Lemma 1 and the following two lemmas, the first of which may be regarded as evident.

LEMMA 2. *Corresponding to each enumerable set E there exists a set G of points* (x_n, y_n) *with* $E \subset G$ *and* $|x_n|, |y_n| < n$, $(n=1, 2, 3, \cdots)$.

LEMMA 3. *Corresponding to each enumerable set E there exists a curve* $\mathfrak{B}_r$ *of the class* $\{\mathfrak{B}_r\}$ *which passes through every point of E.*‡

PROOF. Setting

† It is worthy of note that, by Theorem 2, the existence of a unique limit for $f[\phi(s), \psi(s)]$ as s, $(|s| \leqq r' < r)$, tends to zero for every curve of $\{\mathfrak{B}_r\}$ does *not* imply the existence of (1).

‡ It may well be that this lemma or something like it is known, but we have been unable to locate it in the literature.

$$g_m(w) \equiv 2(-1)^{m+1} \prod_{m \neq \nu = 1}^{\infty} \left(1 - \frac{w^2}{\nu^2}\right) \equiv (-1)^{m+1} \frac{2m^2 \sin \pi w}{\pi w(m^2 - w^2)},$$

we have for $m = 1, 2, 3, \cdots$,

$$(5) \quad \begin{gathered} |g_m(w)| \leqq 2e^{k|w|^2}, \quad \text{where} \quad k = \sum_{\nu=1}^{\infty} 1/\nu^2, \\ g_m(\pm m) = 1, \quad g_m(\pm n) = 0, \quad (m \neq n = 1, 2, 3, \cdots). \end{gathered}$$

We first assume r finite; let $\rho = |r|$ and μ be the greatest integer $\leqq 1/\rho$. Then there exists a σ satisfying the relation

$$(6) \qquad \rho m - 1 > \sigma > 0, \qquad (m = \mu + 1, \mu + 2, \cdots).$$

We define expressions c_n by the formula

$$(7) \qquad c_m = 1/[m^4(\rho m - 1)], \qquad (m = \mu + 1, \mu + 2, \cdots).$$

By Lemma 2 there exists a set G of points (ξ_n, η_n) with $G \supset E$ and $|\xi_n|$, $|\eta_n| < n$, $(n = 1, 2, 3, \cdots)$. Letting $m = \mu + n$, $x_m = \xi_n$, $y_m = \eta_n$, $(n = 1, 2, 3, \cdots)$, we have

$$(8) \qquad |x_m|, \ |y_m| < m - \mu \leqq m, \qquad (m = \mu + 1, \mu + 2, \cdots).$$

From (5), (6), (7), (8), we obtain

$$|c_m x_m g_m(w)|, \ |c_m y_m g_m(w)| \leqq 2e^{k|w|^2}/\sigma m^3,$$

which shows that each of the infinite series

$$(9) \quad F_1(w) \equiv \sum_{m=\mu+1}^{\infty} c_m x_m g_m(w), \qquad F_2(w) \equiv \sum_{m=\mu+1}^{\infty} c_m y_m g_m(w)$$

converges uniformly in any finite region, and accordingly represents an entire function since $g_m(w)$ is entire. Moreover, since G may be assumed to include a point not on either axis, it is evident from the definitions of c_m and $g_m(w)$ that neither $F_1(w)$ nor $F_2(w)$ is a constant. Consequently

$$(10) \quad F_3(w) \equiv -w^4(rw + 1)F_1(w), \quad F_4(w) \equiv -w^4(rw + 1)F_2(w)$$

are entire functions with singularities at $w = \infty$. By means of the transformation

$$(11) \qquad w = 1/(z - r),$$

$F_3(w)$, $F_4(w)$ are transformed respectively into functions $\phi(z)$, $\psi(z)$ which belong to $\{\Gamma_r\}$ and thus determine a curve $\mathfrak{B}_r^*$ of the form (4). Finally [using (11), (10), (9), (7), and (5)] we obtain for $n=\mu+1, \mu+2, \cdots$

$$\phi(r - 1/n) = x_n, \quad \psi(r - 1/n) = y_n, \quad \text{if } r > 0,$$
$$\phi(r + 1/n) = x_n, \quad \psi(r + 1/n) = y_n, \quad \text{if } r < 0,$$

which proves that the curve $\mathfrak{B}_r^*$ passes through each point of G; E being a subset of G, the lemma is established for the case of r finite.

For $r=\infty$, the functions

$$\phi(z) \equiv z^4 \sum_{m=\mu+1}^{\infty} x_m g_m(z)/m^4, \qquad \psi(z) \equiv z^4 \sum_{m=\mu+1}^{\infty} y_m g_m(z)/m^4$$

which belong to $\{\Gamma_\infty\}$, lead to the same conclusion if z is assigned the values $n=\mu+1, \mu+2, \cdots$.

In passing it seems of interest to mention the following corollary.

COROLLARY. *There exists a curve* $\mathfrak{B}_r$ *of the class* $\{\mathfrak{B}_r\}$ *which passes through every point in the plane with rational coordinates.*

From Lemma 3 it is clear that the class $\{\mathfrak{B}_r\}$ has Property L; Theorem 2 then follows by Lemma 1.

4. *The Class of Curves* $\{\mathfrak{C}\}$. Let $F(x, y) \not\equiv 0$ be a real, single-valued function of the real variables x, y which is analytic in some neighborhood of Q and for which $F(0, 0)=0$. Then $F(x, y)=0$ defines a curve $\mathfrak{C}$ through Q. Excluding those curves for which Q is an isolated point, we denote by $\{\mathfrak{C}\}$ the class of all curves $\mathfrak{C}$ which remain. By employing a well known theorem of Weierstrass,† together with an analog of the Puiseux method for algebraic curves, one may readily verify that for each curve $\mathfrak{C}$ of $\{\mathfrak{C}\}$ there exists a neighborhood of Q in which all points of $\mathfrak{C}$ lie on a *finite* number of curves of class $\{\mathfrak{A}\}$. Combining this fact with the proof of Theorem 2 we obtain the following theorem.

THEOREM 4. *The existence of a unique limit for* $f(P)$ *as* $P \to Q$ *on every curve of* $\{\mathfrak{C}\}$ *does not imply the existence of* (1).

† Goursat-Hedrick-Dunkel, *Functions of a Complex Variable*, pp. 233 ff.

5. *The Class of Curves* $\{\mathfrak{D}\}$. Let $\{\mathfrak{D}\}$ denote the class of all curves $\mathfrak{D}$ representable parametrically as

$$x = x(s), \qquad y = y(s), \qquad (0 \leqq s \leqq 1),$$

where $x(s)$ and $y(s)$ have derivatives of all orders and $x(0) = y(0) = 0$.

THEOREM 5. *If* $f[x(s), y(s)]$ *has a unique limit as* s *tends to zero for every curve of* $\{\mathfrak{D}\}$, *the double limit* (1) *exists.*

PROOF. Let S be any set of points having the point Q as a limit point, and let S^* be a subset of points (x_n, y_n) tending to Q such that we have $|x_n|, |y_n| < e^{-1/n^2}$, $(n = 1, 2, 3, \cdots)$. If we set $I_1 \equiv (1/2 \leqq s \leqq 1)$, and $I_n \equiv [1/(n+1) \leqq s \leqq (2n+1)/(2n(n+1))]$, $(n = 2, 3, 4, \cdots)$, then the equations $x(0) = 0$, $x(s) = x_{n+1}$ for s in I_n, define a function with a closed domain which can be extended† to the whole interval $(0 \leqq s \leqq 1)$ in such a way that the extended function $x(s)$ has derivatives of all orders. The function $y(s)$ is defined similarly. The corresponding curve $\mathfrak{D}$ is such that the point $[x(s), y(s)]$ approaches Q through the set S^* as s tends to zero. This proves that $\{\mathfrak{D}\}$ has Property L, and establishes the theorem.

6. *The Class of Curves* $\{\mathfrak{C}\}$. Let $\{\mathfrak{C}\}$ be the class of all curves $\mathfrak{C}$ through Q, each of which has, with respect to a properly chosen system of rectangular coordinates ξ, η with origin at Q, an equation of the form $\eta = \phi(\xi)$, where $\phi(\xi)$ is a single-valued function with a continuous, non-negative, monotonic increasing first derivative in a certain neighborhood of $\xi = 0$ and $\phi'(0) = 0$. For a fixed system ξ, η denote by $x(\xi, \eta)$, $y(\xi, \eta)$ the coordinates of the point (ξ, η) in the original system x, y. Concerning the class of curves $\{\mathfrak{C}\}$ we have the following theorem which is an improvement over Theorem 1 to the extent that $\{\mathfrak{C}\}$ is a proper subclass of the class considered by Clarkson.

THEOREM 6. *If* $f[x(\xi, \phi(\xi)), y(\xi, \phi(\xi))]$ *has a unique limit as* ξ *tends to zero for every curve of* $\{\mathfrak{C}\}$, *the double limit* (1) *exists.*

PROOF. S being any set of points having Q as a limit point one readily sees by Clarkson's reasoning that axes ξ, η can be

† Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63–89, Theorem 1.

so chosen that every closed sector lying in the first quadrant and having the ξ axis as one boundary will contain a subset of S having Q as a limit point. If S has a subset on the ξ axis with Q as a limit point, the curve $\eta=\phi(\xi)\equiv 0$ of class $\{\mathfrak{C}\}$ passes through a subset of S with the limit point Q, and the definition of Property L is satisfied. In the alternative case, we can, by the choice of axes, select a subset S^* of S of points (ξ_n, η_n) tending to Q, such that we have

$$0 < \xi_{n+1} < \xi_n/2, \qquad 0 < \eta_{n+1} < \eta_n/2,$$

$$\eta_n/\xi_n \to 0 \text{ as } n \to \infty, \qquad 0 < 2\eta_{n+1}/\xi_{n+1} < \eta_n/(2\xi_n).$$

From these relations it follows that

$$\frac{2\eta_{n+1}}{\xi_{n+1}} < \frac{\eta_n}{2\xi_n} < \frac{\eta_n - \eta_{n+1}}{\xi_n} < \frac{\eta_n - \eta_{n+1}}{\xi_n - \xi_{n+1}} < \frac{\eta_n}{\xi_n - \xi_{n+1}} < \frac{2\eta_n}{\xi_n};$$

hence $\sigma_n \equiv (\eta_n - \eta_{n+1})/(\xi_n - \xi_{n+1})$ tends monotonically to zero in the strict sense as $n\to\infty$. Consider the sequence of functions $\phi_n(\xi)$ defined as follows. Let $\phi_n(\xi)=\eta_{n+1}+\sigma_n(\xi-\xi_{n+1})$ on the interval $I_n \equiv (\xi_{n+1} \leqq \xi \leqq \xi_n)$ for n odd. For n even, let $\phi_n(\xi)$ be any function on I_n such that $\phi_n(\xi_{n+1})=\eta_{n+1}$, $\phi_n(\xi_n)=\eta_n$, $\phi_n'(\xi_{n+1}+0)=\sigma_{n+1}$, $\phi_n'(\xi_n-0)=\sigma_{n-1}$, and such that $\phi_n'(\xi)$ is continuous and increases monotonically from σ_{n+1} to σ_{n-1} as ξ increases from ξ_{n+1} to ξ_n. That such a function exists is clear from the fact that an arc of an ellipse† can be found whose equation satisfies these conditions.

In the interval $-\xi_1 < \xi < \xi_1$, let $\phi(\xi)=0$ for $-\xi_1 < \xi \leqq 0$, and let $\phi(\xi)=\phi_n(\xi)$ on I_n, $(n=1, 2, 3, \cdots)$. Then it is easily verified that the curve $\eta=\phi(\xi)$ is of class $\{\mathfrak{C}\}$, and by construction it passes through the set S^* as ξ tends to zero through positive values. This completes the proof that $\{\mathfrak{C}\}$ has Property L, and establishes Theorem 6.

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† Such an ellipse is given by the equation

$$[\eta-\eta_{n+2}-\sigma_{n+1}(\xi-\xi_{n+2})]\,[\eta-\eta_n-\sigma_{n-1}(\xi-\xi_n)]-k[\eta-\eta_{n+1}-\sigma_n(\xi-\xi_{n+1})]^2=0,$$

for each $k>(\sigma_{n-1}-\sigma_{n+1})^2/(4(\sigma_{n-1}-\sigma_n)\,(\sigma_n-\sigma_{n+1}))$.