THE RELATIVE CONNECTIVITIES OF SYMMETRIC PRODUCTS*

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1. *Introduction.* The topology of the domain of discontinuity of a finite group of transformations operating on a complex, and, in particular, the topology of symmetric product complexes, has been studied by P. A. Smith† and the author.‡ Following a suggestion made by Morse,§ we obtain in this note explicit formulas for the so-called relative connectivities of the symmetric product of a complex in terms of its mod 2 Betti numbers, and we discuss an application of this result to the theory of critical chords. First, however, we derive a more general result of which the formulas for the relative connectivities of symmetric products is a special case. The methods used here follow closely those of S.

2. *Definitions and Preliminary Theorems.* For proofs or fuller discussion of statements made in this section, the reader is referred to S or R.

Let K be a simplicial n-complex.|| Let T be a topological involution such that (a) T carries m-simplexes of K into m-simplexes of K; (b) if a simplex of K is invariant, it is pointwise invariant.

The invariant simplexes of K form a subcomplex K^0, and the non-invariant simplexes can be grouped in pairs so that each member of a pair is transformed into the other member by T. Thus the m-simplexes of K can be renamed E_m^i, $\overline{E}_m^i$, E_m^{0j}, where $\overline{E}_m^i = TE_m^i$, and E_m^{0j} is a simplex of K^0. If¶ $C = t_i E_m^i$ is a chain of

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† P. A. Smith, *The topology of involutions*, Proceedings of the National Academy of Sciences, (1933), pp. 612–618. (Denoted hereafter by S.)

‡ M. Richardson, *On the homology characters of symmetric products*, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. (Denoted hereafter by R.)

§ M. Morse, *The Calculus of Variations in the Large*, Colloquium Publications of this Society, vol. 18, 1934, p. 191. (Denoted hereafter by M.)

|| Our general topological terminology and notation is that of S. Lefschetz, *Topology*, Colloquium Publications of this Society, vol. 12, 1930.

¶ A repeated index indicates summation.

K, we define TC to be the chain $\overline{C}=t_i\overline{E}_m^i$. The involution T preserves bounding relations.†

We consider only mod 2 topology; all homologies and equations are understood to be homologies and congruences mod 2.

A chain X of K is called invariant if $X=\overline{X}$. In particular, if every simplex occurring in a chain with a non-zero coefficient belongs to K^0, we attach a zero to the chain-symbol, as X^0. If no simplex of K^0 occurs in a chain with a non-zero coefficient, we attach an asterisk to the chain-symbol, as X^*. Every invariant chain can be written in the form $X^*+\overline{X}^*+X^0$. If an invariant cycle Γ is the boundary of an invariant chain, we write $\Gamma\cong 0$. These *special* homologies obey the same formal rules as ordinary homologies.

We choose a base for homology of type Γ^i, $\overline{\Gamma}^i$, D^i for each dimension.‡ We consider only the case in which (A) $D_m^j+\overline{D}_m^j\cong 0$ for every $m>0$ and every j. In this case we can and do replace the D_m^i in the base by invariant cycles§ ${}^i\Delta_m$.

We now construct the sequences

$$(1) \qquad {}^i\Delta_m^m,\ {}^i\Delta_{m-1}^m,\ \cdots,\ {}^i\Delta_r^m, \qquad (r=r(m,i)\geqq -1),$$

where ${}^i\Delta_q^m={}^iX_q^m+{}^i\overline{X}_q^m$, and $F({}^iX_q^m)={}^i\Delta_{q-1}^m$, (for $q=m, m-1, \cdots, r+1$), and ${}^i\Delta_r^m={}^iX_r^m+{}^i\overline{X}_r^m+{}^iX_r^{0m}$, (for $r\geqq 0$), where the ${}^iX_r^{0m}$ are cycles.‖ We consider only the case where (B) the cycles ${}^iX_r^{0m}$ are independent with respect to homologies on K^0. We shall need the following lemmas.¶

(2) *If* $C+\overline{C}+C^0\cong 0$, *then* $C^0\sim 0$ *on* K^0.

(3) *The cycles* $\Gamma_q^i+\overline{\Gamma}_q^i$, ${}^i\Delta_q^m$, $(q>0)$, *are independent with respect to* $\cong$.

(4) *If* (A) *and* (B) *hold, every cycle of the form* $C_q+\overline{C}_q$, $(q>0)$, *is* $\cong$ *to a linear combination of cycles* $\Gamma_q^i+\overline{\Gamma}_q^i$, ${}^i\Delta_q^m$.

With the simplexes E_m^i and $\overline{E}_m^i$ we associate† a simplex e_m^i, and we write $\wedge E_m^i=\wedge\overline{E}_m^i=e_m^i$. If $C=t_iE_m^i$, we define $\wedge C$ to be the chain $c=t_ie_m^i$. The totality of simplexes e_m^i constitutes

† R, §1.

‡ S, §1.

§ S, p. 614.

‖ S, p. 614.

¶ S, pp. 613–615.

† The material in this paragraph is fully discussed in R, §2.

an n-complex $k = \wedge K$, say. In particular, the simplexes $e_m^{0i} = \wedge E_m^{0i}$ constitute a subcomplex $k^0 = \wedge K^0$, say. If $e = \wedge E$ is a simplex of k, we write $\wedge' e = E + \overline{E}$. If $c = t_i e_m^i$, we define $\wedge' c$ to be the chain $t_i \wedge' e_m^i$. Both $\wedge$ and $\wedge'$ preserve bounding relations. We shall use large or small letters for chains of K or k, respectively. In particular, a symbol like x^0 will denote a chain of k^0, and a symbol like x^* will denote a chain in which no cell of k^0 occurs with a non-zero coefficient.

3. *The Topology of* k mod k^0. We shall now determine the Betti numbers $R_q(k; k^0, 2)$. A chain whose boundary is a chain of k^0, that is, a cycle mod k^0, shall be called a relative cycle.

(5) *If* $c + x^0 \to 0$, *then* c *is a relative cycle.*

Proof. Let $F(c) = y^* + y^0$ and $F(x^0) = z^0$. Since $F(c + x^0) = y^* + y^0 + z^0 = 0$, we have $y^* = 0$. Hence $c \to 0$ mod k^0.

(6) *If* $c + x^0 \sim 0$, *then* $c \sim 0$ mod k^0.

Proof. There exists a chain d such that $d \to c + x^0$. Thus $d \to c$ mod k^0.

(7) *If* γ *is a relative cycle, then* $\wedge' \gamma$ *is a cycle.*

Proof. Since $\gamma \to x^0$, we have $\wedge' \gamma \to \wedge' x^0 = X^0 + \overline{X}^0 = 2X^0 = 0$.

(8) *If* $\gamma \sim 0$ mod k^0, *then* $\wedge' \gamma \cong 0$.

Proof. Since there exists a chain c such that $c \to \gamma + x^0$, we have $\wedge' c \to \wedge' \gamma + \wedge' x^0$. But $\wedge' x^0 = 0$. It is obvious that $\wedge' c$ and $\wedge' \gamma$ are invariant.

(9) *If* c *is a relative cycle, we can write* $\wedge' c$ *in the form* $C + \overline{C}$ *where* $\wedge C = c$.

Proof. Let $c = t_i e_m^i + u_i e_m^{0i}$. We have only to let $C = t_i E_m^i + u_i E_m^{0i}$.

(10) *If* $C + \overline{C} \cong 0$, *then* $\wedge C$ *is a relative cycle and* $\wedge C \sim 0$ mod k^0.

Proof. By hypothesis, $H + \overline{H} \to C + \overline{C}$. Let $F(H) = C + X$. Then $X + \overline{X} = 0$. Hence $X = X^* + \overline{X}^* + X^0$. Therefore,

$$\wedge H \to \wedge C + \wedge X = \wedge C + 2 \wedge X^* + \wedge X^0 = \wedge C + \wedge X^0.$$

Thus $\wedge C + \wedge X^0 \to 0$, and by (5), $\wedge C$ is a relative cycle. Since $\wedge H \to \wedge C + \wedge X^0 \sim 0$, we have $\wedge C \sim 0$ mod k^0, by (6).

(11) *For* $q \geqq r(m, i)+1$, $\wedge({}^iX_q^m)$ *is a relative cycle, say* ${}^i\xi_q^m$, *and* $\wedge'({}^i\xi_q^m) = {}^i\Delta_q^m$.

Proof. If $q \geqq r(m, i)+2$, then, since ${}^iX_q^m \rightarrow {}^i\Delta_{q-1}^m$, we have

$$\wedge({}^iX_q^m) \rightarrow \wedge({}^i\Delta_{q-1}^m) = \wedge({}^iX_{q-1}^m + {}^i\overline{X}_{q-1}^m) = 0.$$

Therefore, $\wedge({}^iX_q^m) = {}^i\xi_q^m$ is an absolute cycle. If $q = r(m, i)+1$, we have

$$\begin{aligned}\wedge({}^iX_q^m) \rightarrow \wedge({}^i\Delta_{q-1}^m) &= \wedge({}^iX_r^m + {}^i\overline{X}_r^m + {}^iX_r^{0m}) \\ &= \wedge({}^iX_r^{0m}) = 0 \bmod k^0.\end{aligned}$$

Thus, in this case, ${}^i\xi_q^m = \wedge({}^iX_q^m)$ is a relative cycle. In either case we have $\wedge'({}^i\xi_q^m) = {}^iX_q^m + {}^i\overline{X}_q^m = {}^i\Delta_q^m$.

Let $\gamma_q^i = \wedge\Gamma_q^i$.

(12) *The relative cycles* γ_q^i, ${}^i\xi_q^m$, $(q \geqq r(m, i)+1 > 0)$, *are independent with respect to homology* mod k^0.

Proof. Suppose there were a non-trivial homology

$$x_{im}{}^i\xi_q^m + y_i\gamma_q^i \sim 0 \bmod k^0.$$

By (8) we have $\wedge'(x_{im}{}^i\xi_q^m + y_i\gamma_q^i) \cong 0$. Thus, by (11),

$$x_{im}{}^i\Delta_q^m + y_i(\Gamma_q^i + \overline{\Gamma}_q^i) \cong 0,$$

contradicting (3).

(13) *Every relative q-cycle of k is homologous* mod k^0 *to a linear combination of the* γ_q^i *and* ${}^i\xi_q^m$, $(q \geqq r(m, i)+1 > 0)$.

Proof. Let γ be an arbitrary relative q-cycle of k. Let $\wedge'\gamma = \Gamma + \overline{\Gamma}$, where $\wedge\Gamma = \gamma$, by (9). By (7), $\wedge'\gamma$ is a cycle. Therefore, by (4),

$$\Gamma + \overline{\Gamma} \cong x_i(\Gamma_q^i + \overline{\Gamma}_q^i) + y_{im}{}^i\Delta_q^m. \tag{14}$$

Now we shall show that $y_{im} = 0$ whenever $r(m, i) = q$. For, if some $y_{im} \neq 0$, then (14) would be of the form

$$Y + \overline{Y} + z_{im}({}^iX_q^m + {}^i\overline{X}_q^m + {}^iX_q^{0m}) \cong 0,$$

where some $z_{im} \neq 0$. This implies that $z_{im}{}^iX_q^{0m} \sim 0$ on K^0, by (2). But this contradicts (B). Thus, (14) has the form

$$\Gamma + \overline{\Gamma} + x_i(\Gamma_q^i + \overline{\Gamma}_q^i) + y_{im}({}^iX_q^m + {}^i\overline{X}_q^m) \cong 0,$$

where $q \geqq r(m, i)+1$. Let $C=\Gamma+x_i\Gamma_q^i+y_{im}{}^iX_q^m$. Then $C+\overline{C}\cong 0$, and, by (10), we have $\wedge C \sim 0 \bmod k^0$, or

$$\gamma + x_i\gamma_q^i + y_{im}{}^i\xi_q^m \sim 0 \bmod k^0.$$

This proves the theorem.

By (12) and (13), the relative cycles ${}^i\xi_q^m$, γ_q^i, $(q \geqq r(m, i)+1 > 0)$, constitute a base for relative q-cycles of k with respect to homology mod k^0. Let R_q^Γ be the number of cycles Γ_q^i, and let Q_q be the number of q-cycles ${}^i\Delta_q^m$ satisfying the relation $r(m, i)+1 \leqq q$. We have proved the following theorem.

THEOREM 1. *If the hypotheses* (a), (b), (A), *and* (B) *are fulfilled, then* $R_q(k; k^0, 2)=R_q^\Gamma+Q_q, (q>0)$.

4. *Symmetric Products.* Let $K_{2n}=K_n\times K_n$ be the complex K of the preceding sections. Let T be the involution which interchanges the points $P\times Q$ and $Q\times P$ of K_{2n}. A simplicial subdivision of K_{2n} satisfying (a) and (b) of §2 can be found.† Of course, $\wedge K_{2n}=k_{2n}$ is the 2-fold symmetric product of K_n. We can choose bases for homology on K_{2n} of the Γ, $\overline{\Gamma}$, ${}^i\Delta$ type here required.‡ The cycles ${}^i\Delta_q$ occur only in even dimensions. It has been shown that the sequences (1) can be constructed so that $r(2h, i)=h$ for all i, and so that (A) and (B) are fulfilled.§ Therefore we may apply Theorem 1.

Now let $R_{2s}^\Delta=R_s(K_n, 2)$ for $s\leqq n$ and $R_{2s}^\Delta=0$ for $s>n$. Then it is easily seen that $Q_1=0$ and

$$Q_q = R_{2t}^\Delta + R_{2(t+1)}^\Delta + \cdots + R_{2(q-1)}^\Delta, \qquad (q > 1),$$

where $t=[(q+1)/2]$, since the lowest dimension $2m$ to yield cycles ${}^i\Delta_q^{2m}$ is either $2m=q$ or $2m=q+1$. Thus by Theorem 1, we have the following result.

THEOREM 2. *For the symmetric product* k_{2n} *of* K_n *we have*

$$R_1(k_{2n}; k_n^0, 2) = R_1^\Gamma$$

$$R_q(k_{2n}; k_n^0, 2) = R_q^\Gamma + R_{2t}^\Delta + R_{2(t+1)}^\Delta + \cdots + R_{2(q-1)}^\Delta, \quad (q > 1),$$

where $t=[(q+1)/2]$, *and where*

† R, §5.

‡ R, p. 57.

§ R, pp. 64–65.

$$\overset{\Gamma}{R}_q = \frac{1}{2}[R_q(K_{2n}, 2) - \overset{\Delta}{R}_q]$$

if q is even, and

$$\overset{\Gamma}{R}_q = \frac{1}{2} R_q(K_{2n}, 2)$$

if q is odd.

Of course, if K_n is connected, so is k_{2n}; hence $R_0(k_{2n}; k_n^0, 2) = 0$ in this case. The numbers $R_q(k_{2n}; k_n^0, 2)$ have been called relative connectivities by Morse,† who proved that they are finite.‡ This result is of course implied by our formulas.

Example 1. Let K_n be an n-sphere. Then $\overset{\Gamma}{R}_n = \overset{\Delta}{R}_{2n} = 1$, while all the other $\overset{\Gamma}{R}$'s and $\overset{\Delta}{R}$'s, are zero. From our formulas, we obtain for the relative connectivities of k_{2n},

$$\begin{aligned}
R_0 &= R_1 = \cdots = R_{n-1} = 0,\\
R_n &= \overset{\Gamma}{R}_n = 1,\\
R_{n+1} &= \overset{\Delta}{R}_{2(q-1)} = \overset{\Delta}{R}_{2n} = 1,\\
R_{n+2} &= \overset{\Delta}{R}_{2(q-2)} = \overset{\Delta}{R}_{2n} = 1,\\
&\cdots\cdots\cdots\cdots\\
R_{2n} &= \overset{\Delta}{R}_{2t} = \overset{\Delta}{R}_{2n} = 1.
\end{aligned}$$

The values of the relative connectivities for this example were worked out by Morse§ by special methods involving the critical chords of an n-ellipsoid.

Example 2. Let K_n be an orientable surface of genus p. Then the relative connectivities of the symmetric product k_{2n} are $R_0 = 0$, $R_1 = 2p$, $R_2 = 2p^2 + p + 1$, $R_3 = 2p + 1$, $R_4 = 1$.

5. *Application to the Theory of Critical Chords.*‖ The chief results concerning critical chords are as follows. Let R be a regular, analytic Riemannian n-manifold lying in a euclidean $(n+1)$-space, such that R is homeomorphic to a simplicial n-complex K_n. Then the symmetric product of R is evidently homeo-

† M, p. 182.

‡ M, pp. 182–183.

§ M, Theorem 11.3, p. 191.

‖ For definitions and proofs required in this section see M, pp. 181–191.

morphic to the symmetric product k_{2n} of K_n. Let $R_0, R_1, \cdots, R_{2n}$ be the relative connectivities of k_{2n}. Then the sums M_i of the type numbers of the critical sets of chords of R and the numbers R_i satisfy the relations

$$M_0 \geqq R_0, M_0 - M_1 \leqq R_0 - R_1, M_0 - M_1 + M_2 \geqq R_0 - R_1 + R_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M_0 - M_1 + \cdots + (-1)^{2n} M_{2n} = R_0 - R_1 + \cdots + (-1)^{2n} R_{2n}. \dagger$$

A simple corollary of this theorem is this: *If the critical chords of R are all non-degenerate, there exist at least R_i such chords of index‡ i.*

Our Theorem 2 enables us to obtain the values of the relative connectivities R_i of k_{2n} when the mod 2 Betti numbers of R are known. Thus the above theorem and its corollary can be used to obtain numerical information concerning the critical chords of any R whose mod 2 Betti numbers are known. This makes available a wide class of examples. For instance, the corollary of M, p. 191 follows at once from the above corollary and our Example 1, §4.

As a further example, let R be any regular, analytic image of an orientable surface of genus p. Then, from Example 2, §4, and the above corollary, we obtain the result that, if the extremal chords of R are all non-degenerate, then, among these extremal chords there must be $2p^2+5p+3$ extremal chords of the following description: $2p$ extremal chords of index 1, $2p^2+p+1$ extremal chords of index 2, $2p+1$ extremal chords of index 3, and 1 extremal chord of index 4. In the degenerate case, the same result holds provided each critical set of chords is counted according to its type numbers.

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† M, Theorem 11.1, p. 185.
‡ M, p. 185.