# THE BLOCH CONSTANT $\mathfrak{A}$ FOR A SCHLICHT FUNCTION*

BY R. M. ROBINSON

The following theorem† is due to Bloch.

*There is an absolute positive constant P with the following property. Let $f(x)$ be regular for $|x|<1, f'(0)=1$. Then $y=f(x)$ maps the circle $|x|<1$ on a region (in a Riemann surface over the y plane) containing a circle of radius P in a single sheet.*

(Without the condition $f'(0)=1$ there is a circle of radius $P|f'(0)|$.)

Landau‡ defines three absolute constants connected with this theorem. $\mathfrak{B}$ is the upper bound of the $P$ which satisfy the theorem as stated. $\mathfrak{L}$ is the upper bound of the $P$ if we require only that there be a circle of radius $P$ in the $y$ plane each point of which is covered by some sheet of the map. $\mathfrak{A}$ is the corresponding bound if $f(x)$ is given as schlicht (that is, $f(x_1)\neq f(x_2)$ for $x_1\neq x_2$, so that the map has only one sheet).

We have clearly $\mathfrak{B}\leqq\mathfrak{L}\leqq\mathfrak{A}$. The chief object of the paper by Landau is to give as close lower and upper bounds as possible for these three constants. He proves $0.39<\mathfrak{B}<0.56$, $0.43<\mathfrak{L}<0.56$, and $\mathfrak{A}>0.56$ (so that $\mathfrak{L}<\mathfrak{A}$). However, as an upper bound for $\mathfrak{A}$ he obtains no new result, but mentions a result of Szegö $\mathfrak{A}\leqq\pi/4$, as the best result which he knows. This follows from consideration of the function

$$y=\frac{1}{2}\log\frac{1+x}{1-x}=x+\cdots,$$

which maps the circle $|x|<1$ on the strip $|\Im y|<\pi/4$.

A better bound than $\pi/4$ can be obtained by mapping $|x|<1$ on as simple a region as a circle slit along a radius from the center to the circumference (Theorem 1). Still better bounds may be

* Presented to the Society, October 27, 1934.

† For a proof and further references, see a paper by Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634.

‡ Loc. cit.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

obtained by using several slits extending along radii part way to the center (Theorems 2, 3, 4). Using this method I shall show $\mathfrak{A}<0.66$.

LEMMA 1. *If*

$$k(x) = \frac{x}{(1-x)^2}, \qquad (|x|<1),$$

*and $K$ is the inverse function, and we put*

$$p = p(r) = \frac{4r}{(1+r)^2}, \qquad (0<r<1),$$

*then $z=K(pk(x))$ maps the circle $|x|<1$ on the circle $|z|<1$ slit from $-1$ to $-r$.*

PROOF. This is an immediate consequence of the fact that $k(x)$ maps the circle $|x|<1$ on a plane cut from $-1/4$ to $-\infty$.*

THEOREM 1. *An upper bound for $\mathfrak{A}$ is given by the formula*

$$\mathfrak{A} \leqq \frac{3+2(2)^{1/2}}{8} (<0.729).$$

PROOF. If we combine the transformations

$$z = K(pk(x)) = px + \cdots, \quad y = \frac{z+r}{rz+1} = r + (1-r^2)z + \cdots,$$

we have a transformation $y=f(x)$ which maps the circle $|x|<1$ on the circle $|y|<1$ slit from $-1$ to 0. We have

$$f'(0) = p(1-r^2) = \frac{4r(1-r)}{1+r}.$$

We choose $r$ so that $f'(0)$ is a maximum;

$$r = (2)^{1/2}-1, \qquad f'(0) = 4((2)^{1/2}-1)^2.$$

Since the map contains no circle of radius greater than 1/2, we have

$$\mathfrak{A}f'(0) \leqq \frac{1}{2}, \qquad \mathfrak{A} \leqq \frac{3+2(2)^{1/2}}{8}.$$

* See Landau's book, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 1929, pp. 112–113.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

LEMMA 2. *If a function $F(x)$, regular for $|x|<1$ and such that $F(0)=0$, maps the circle $|x|<1$ on a plane region $R$, then $F(x^n)$ maps $|x|<1$ on $n$ copies of $R$ connected by a branch point at the origin, so that $[F(x^n)]^{1/n}$ maps $|x|<1$ on a plane region which is $n$-fold symmetric with respect to the origin.*

This known result is so evident as to require no proof.

THEOREM 2. *An upper bound for $\mathfrak{A}$ is given by the formula*

$$\mathfrak{A} \leqq \frac{3}{8}(6)^{1/3}(< 0.682).$$

PROOF. In Lemma 2, put $F(x)=K(pk(x))$ and $n=3$. This leads to the result that $y=f(x)=[K(pk(x^3))]^{1/3}$ maps the circle $|x|<1$ on the circle $|y|<1$ with three cuts along equally spaced radii to within a distance $(r)^{1/3}$ of the origin. For this function $f'(0)=(p)^{1/3}$. If we choose $r=1/8$, then the three cuts extend half way to the origin (Fig. 1). The largest circles that can be

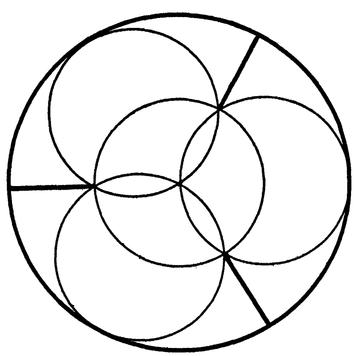

FIG. 1.

drawn in the map are of radius 1/2, there being four of this size. Hence

$$\mathfrak{A}(p)^{1/3} \leqq \frac{1}{2}, \qquad \mathfrak{A} \leqq \frac{3}{8}(6)^{1/3}.$$

LEMMA 3. *If*

$$p_1 = p(r_1), \qquad p_2 = p(r_2), \qquad (0 < r_1 < 1,\ 0 < r_2 < 1),$$

*and $p$ is determined from the relation*

$$\frac{1}{pp_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1,$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

*then the function* $F(x)=K(p_1k(-K(pk(x))))$ *maps* $|x|<1$ *on a unit circle cut from* $-1$ to $-r_1$ *and from* 1 *to* $r_2$. *For this function*

$$\frac{1}{|F'(0)|}=\frac{1}{4}\left(r_1+\frac{1}{r_1}+r_2+\frac{1}{r_2}\right).$$

PROOF. $K(pk(x))$ maps $|x|<1$ on a unit circle cut from $-1$ to $-r$. For $-K(pk(x))$ the cut is from 1 to $r$. Proceeding to $F(x)$ inserts a second cut from $-1$ to $-r_1$ as required, but distorts the first cut. The symmetry of the figure with respect to the real axis shows that the cut will still lie along the real axis. Its end point $r$ is taken to $K(p_1k(r))$, which is equal to $r_2$, as required, if $p_1k(r)=k(r_2)$. Since

$$\frac{1}{k(r)}=4\left(\frac{1}{p(r)}-1\right),$$

this becomes

$$\frac{1}{p_1}\,4\left(\frac{1}{p}-1\right)=4\left(\frac{1}{p_2}-1\right),\qquad \frac{1}{pp_1}=\frac{1}{p_1}+\frac{1}{p_2}-1.$$

Since $|F'(0)|=pp_1$, this gives the last statement of the lemma if we put in the values of $p_1$ and $p_2$.

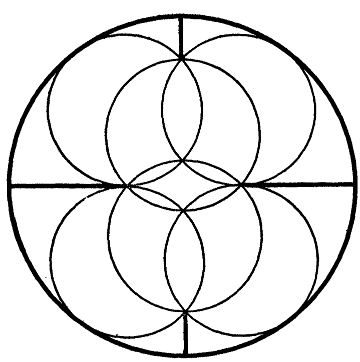

FIG. 2.

THEOREM 3. *An upper bound for* $\mathfrak{A}$ *is given by the formula*

$$\mathfrak{A}^2\leqq\frac{(1+v^2)^4(1+v-v^2)}{16v^2(1+v)^3(1-v)}\quad for\quad (2)^{1/2}-1\leqq v<1.$$

*Putting* $v=0.55$, *we find* $\mathfrak{A}<0.666$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

PROOF. We shall map the circle $|x|<1$ on a unit circle with cuts from 1 to $u$, $-1$ to $-u$, $i$ to $vi$, $-i$ to $-vi$ (Fig. 2). The quantities $u$, $v$ are to be so related that each of the four circles tangent to the real axis at $\pm u$, and tangent to the unit circle, shall pass through one of the points $\pm vi$. Let the radius of these circles be $\rho$. Then

$$u^2 + \rho^2 = (1 - \rho)^2, \qquad u^2 + (v - \rho)^2 = \rho^2,$$

from which we find

$$u^2 = \frac{v(1 - v)}{1 + v}, \qquad \rho = \frac{1 + v^2}{2(1 + v)}.$$

If $v \geqq (2)^{1/2} - 1$, the maximum circles in the map are six of radius $\rho$, the four mentioned, and two through $\pm u$ and one of the points $\pm vi$. (For $v = (2)^{1/2} - 1$, $u = (2)^{1/2} - 1$ and four of the circles are tangent to both axes, the other two coinciding at the center.) Hence if $f(x)$ is the mapping function, $\mathfrak{A}|f'(0)| \leqq \rho$. To accomplish the mapping, we first map $|x|<1$ on a unit circle with cuts from $-1$ to $-v^2$ and from 1 to $u^2$ by the function $F(x)$ of Lemma 3 (putting $r_1 = v^2$, $r_2 = u^2$). For this function

$$\frac{1}{|F'(0)|} = \frac{1}{4}\left(v^2 + \frac{1}{v^2} + u^2 + \frac{1}{u^2}\right) = \frac{(1 + v^2)^2(1 + v - v^2)}{4v^2(1 + v)(1 - v)}.$$

We then put $f(x) = [F(x^2)]^{1/2}$ for the final mapping function. Since $|f'(0)|^2 = |F'(0)|$, we have

$$\mathfrak{A}^2 |F'(0)| \leqq \rho^2,$$

which gives the theorem.

THEOREM 4. *An upper bound for* $\mathfrak{A}$ *is given by the formula*

$$\mathfrak{A}^3 \leqq \frac{9}{4}\left(\frac{1}{a^6} + \frac{1}{8}\right),$$

*where* $a = 1 + (b^2 + 2b + 2)^{1/2}$, $b = [2(3)^{1/2} - 3]^{1/2}$. *Hence* $\mathfrak{A} < 0.658$.

PROOF. We shall map on a region constructed as follows. Start with a circle of radius $2\rho$ about the origin, with three cuts half way to the center (the dotted circle in Fig. 3). It contains four circles of radius $\rho$, as in Fig. 1. The three cuts are extended outward, and three more cuts are drawn outward from the cir-

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

cumference. Six circles of radius $\rho$ are then pushed as far as possible toward the center in the six sections. Consider, for example, the circle in the upper left section. We may slide it along the real axis toward the origin until it strikes the point $-\rho+(3)^{1/2}\rho i$, the end of one of the cuts. Its center then reaches the position

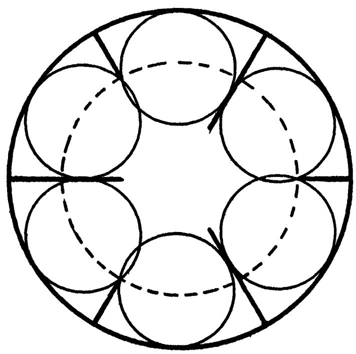

FIG. 3.

$-(b+1)\rho+\rho i$, at a distance $\rho(b^2+2b+2)^{1/2}$ from the origin. A circle of radius $a\rho$ will then include the six circles. We choose $\rho=1/a$, so that the latter circle is the unit circle. We thus are to map on a unit circle with three cuts extending to within a distance $1/a$ and three extending to within a distance $2/a$ from the origin. This is accomplished by taking the $F(x)$ of Lemma 3 with $r_1=1/a^3$, $r_2=8/a^3$ and then putting $y=f(x)=[F(x^3)]^{1/3}$. Now

$$\frac{1}{|F'(0)|}=\frac{1}{4}\left(r_1+\frac{1}{r_1}+r_2+\frac{1}{r_2}\right)=\frac{9}{4}\left(\frac{1}{a^3}+\frac{a^3}{8}\right).$$

Since $|f'(0)|^3=|F'(0)|$ and the map contains no circle of radius greater than $\rho=1/a$, we have

$$\mathfrak{A}\,|f'(0)|\leqq\frac{1}{a},\qquad \mathfrak{A}^3\,|F'(0)|\leqq\frac{1}{a^3},$$

which gives the theorem.

THE UNIVERSITY OF CALIFORNIA AT BERKELEY

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use