DIFFERENTIATION OF SEQUENCES*

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Conditions under which one may differentiate term by term a convergent sequence of functions are to be found in the literature.† Some of these conditions are obtained as corollaries of theorems about termwise integration, and consequently contain in the hypothesis the assumption that the derived sequence converges. Without this assumption, sufficient conditions for termwise differentiability may be obtained from the fundamental theorem on reversing the order of iterated limits. This possibility is mentioned by Hobson (vol. 2, p. 336), but he does not obtain the theorems of the present paper.

By this means it may be shown that the equicontinuity of the functions of the derived sequence is sufficient for termwise differentiability, but the condition of equicontinuity is stronger than necessary for mere convergence, since it implies also that the derivatives are continuous and converge uniformly to a continuous function. This paper is concerned with weaker conditions, for which the names *normal*, *uniform*, *almost normal*, *almost uniform* are used. These conditions, like that of equicontinuity, are related also to the question of the compactness of sets of functions for different types of convergence. The procedure that will be followed is to state the principal lemma, define the different conditions, and then show their relations, first to compactness, and then to termwise differentiation.

LEMMA. *If the sequence of functions* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on the closed interval* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist at the point* x_0 *of* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x_0)$ *exist at* x_0 *and be the limit of the sequence of derivatives* (ii) $\{f_n'(x_0)\}$ *is that for every* $\epsilon>0$ *there exists a* $\delta>0$ *such that if* $|x-x_0|<\delta$, *then*

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right| < \epsilon \tag{1}$$

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† Hobson, *The Theory of Functions of a Real Variable*, 2nd ed., vol. 2, pp. 332–338.

for almost all n (that is, for all except a finite number of values of n).

This is a special case of a general theorem on reversing the order of iterated limits,* which may be stated as follows:

If $\lim_{x\to x_0}F_n(x)=F_n$, *and* $\lim_{n\to\infty}F_n(x)=F(x)$ *for* $x\neq x_0$, *then* $\lim_{x\to x_0}\lim_{n\to\infty}F_n(x)=\lim_{n\to\infty}\lim_{x\to x_0}F_n(x)$ *if and only if for every* $\epsilon>0$ *there exists a* $\delta>0$ *such that if* $|x-x_0|<\delta$, *then* $|F_n(x)-F_n|<\epsilon$ *for almost all n.*

The lemma follows from this if we let

$$F_n(x)=\frac{f_n(x)-f_n(x_0)}{x-x_0}\quad\text{and}\quad F_n=f_n'(x_0).$$

Definitions. The sequence (i) $\{f_n(x)\}$ is said to be *normal*† at the point x_0 of $[a, b]$ if for every $\epsilon>0$ there exists a $\delta>0$ and an N such that if $|x-x_0|<\delta$ and $n>N$, then $|f_n(x)-f_n(x_0)|<\epsilon$.

The sequence (i) is said to be *almost normal* at x_0 if for every $\epsilon>0$ there exists a $\delta>0$ such that if $|x-x_0|<\delta$, then $|f_n(x)-f_n(x_0)|<\epsilon$ for almost all n.

The sequence (i) is said to be *uniform* at x_0 if for every $\epsilon>0$ there exists a $\delta>0$ and an N such that if $|x-x_0|<\delta$ and $m>N$ and $n>N$, then $|f_m(x)-f_m(x_0)-f_n(x)+f_n(x_0)|<\epsilon$.

The sequence (i) is said to be *almost uniform* at x_0 if for every $\epsilon>0$, there exists a $\delta>0$ such that if $|x-x_0|<\delta$, then $|f_m(x)-f_m(x_0)-f_n(x)+f_n(x_0)|<\epsilon$ for all m, n sufficiently large.

Note that in the definitions of *normal* and *uniform* the value of N is selected in advance of the value of x. Note also these relationships: if a sequence is equicontinuous, it is normal, and if it is normal, it is both uniform and almost normal, and if it is either uniform or almost normal, it is almost uniform. Note also that (i) is uniform if and only if the double sequence (ii) $\{f_m(x)-f_n(x)\}$ is normal, and (i) is almost uniform if and only if (ii) is almost normal, when the definitions are extended to double sequences.

The following results concerning normal sequences will be restated from Hahn and Carathéodory:

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* Hobson, vol. 1, p. 385; Hahn, *Theorie der Reellen Funktionen*, 1st ed., pp. 288, 294.

† Hahn, 2nd ed., p. 229; Carathéodory, Mathematische Annalen, vol. 101 (1929), p. 515.

A necessary and sufficient condition that every subsequence of (i) $\{f_n(x)\}$ *contain in turn a subsequence which converges continuously (that is, uniformly to a continuous function) on* $[a, b]$ *is that* (i) *be normal at every point of* $[a, b]$ *and bounded in* n *for every* x. *If* (i) *converges at* x_0, *it converges continuously at* x_0 *if and only if it is normal at* x_0.

We shall prove analogous results for the other conditions.

THEOREM 1. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *it is almost uniform at every point of* $[a, b]$. *Furthermore, it converges uniformly on* $[a, b]$ *if and only if it is uniform at every point of* $[a, b]$; *and the limit function* $f(x)$ *is continuous at a point* x_0 *of* $[a, b]$ *if and only if the sequence* (i) *is almost normal at* x_0.

Since (i) converges on $[a, b]$, given $\epsilon > 0$ and two points x and x_0 of $[a, b]$, there exists an N such that for all $m > N$, $n > N$

$$|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon. \tag{2}$$

Hence (i) is almost uniform at every point x_0 of $[a, b]$.

Now if (i) is uniform at x_0, given $\epsilon > 0$ select $\delta > 0$ and N' such that if $|x - x_0| < \delta$ and $m > N'$, $n > N'$, then (2) holds. Since (i) converges at x_0, select $N > N'$ such that if $m > N$, $n > N$, then

$$|f_m(x_0) - f_n(x_0)| < \epsilon. \tag{3}$$

Adding (2) and (3), we have for $|x - x_0| < \delta$, $m > N$, $n > N$,

$$|f_m(x) - f_n(x)| < 2\epsilon. \tag{4}$$

Hence the convergence is locally uniform* at every point x_0 of $[a, b]$. Since $[a, b]$ is a closed interval, the convergence is uniform on $[a, b]$.

Conversely, if (1) converges uniformly on $[a, b]$ then for every $\epsilon > 0$ an N exists such that for any two points x and x_0 of $[a, b]$, we have, for $m > N$, $n > N$,

$$|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon. \tag{5}$$

Hence the sequence is uniform at every point x_0 of $[a, b]$.

Next, if (1) is almost normal at x_0, given $\epsilon > 0$ select $\delta > 0$ such that if $|x - x_0| < \delta$, then for almost all n

$$|f_n(x) - f_n(x_0)| < \epsilon. \tag{6}$$

* Hahn, 2nd ed., pp. 214, 216.

Since (1) converges at x and x_0, we have for almost all n

$$(7) \qquad |f_n(x_0) - f(x_0)| < \epsilon, \qquad |f_n(x) - f(x)| < \epsilon.$$

Adding (6) and (7), we have, if $|x - x_0| < \delta$,

$$(8) \qquad |f(x) - f(x_0)| < 3\epsilon.$$

Hence $f(x)$ is continuous at x_0. Conversely, if $f(x)$ is continuous at x_0, given $\epsilon > 0$ select $\delta > 0$ such that if $|x - x_0| < \delta$, we have

$$(9) \qquad |f(x_0) - f(x)| < \epsilon.$$

Since (1) converges at x and x_0, then (7) holds for almost all n. Adding (7) and (9) we have, if $|x - x_0| < \delta$, for almost all n

$$(10) \qquad |f_n(x_0) - f_n(x)| < 3\epsilon.$$

Hence (1) is almost normal at x_0.

THEOREM 2. *If the sequence* (i) $\{f_n(x)\}$ *is almost uniform at every point of* $[a, b]$ *and bounded in* n *for every* x, *then every subsequence of* (i) *contains in turn a subsequence* (ii) $\{f_{n_k}(x)\}$ *which converges on* $[a, b]$ *to some function* $f(x)$. *Furthermore, if* (i) *is uniform at every point of* $[a, b]$, *the convergence of* (ii) *is uniform on* $[a, b]$, *and if* (i) *is almost normal at every point of* $[a, b]$, *the limit function* $f(x)$ *of* (ii) *is continuous on* $[a, b]$.

Since (i) is bounded in n for every x, from every subsequence of (i) a subsequence (ii) $\{f_{n_k}(x)\}$ may be chosen* which converges at every rational point of $[a, b]$. Now let x_0 be any irrational point of $[a, b]$. Since (i) is almost uniform at x_0, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$(11) \qquad |f_{n_k}(x) - f_{n_k}(x_0) - f_{n_l}(x) + f_{n_l}(x_0)| < \epsilon$$

for all k, l sufficiently large. Choose x rational. Then (ii) converges at x, and for k and l sufficiently large we have

$$(12) \qquad |f_{n_k}(x) - f_{n_l}(x)| < \epsilon.$$

From (11) and (12), we get

$$(13) \qquad |f_{n_k}(x_0) - f_{n_l}(x_0)| < 2\epsilon.$$

Hence (ii) converges also at x_0, and therefore at every point of $[a, b]$. The second part of the theorem follows from Theorem 1.

* Hahn, 2nd ed., p. 230.

THEOREM 3. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on the closed interval* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist on* $[a, b]$ *and the sequence of derivatives* (ii) $\{f_n'(x)\}$ *is uniform at a point* x_0 *of* $[a, b]$, *then the derivative* $f'(x_0)$ *exists at* x_0, *and the derived sequence* (ii) *converges to* $f'(x_0)$ *at* x_0.

Given $\epsilon>0$, select $\delta>0$ and N in accordance with the condition that (ii) is uniform at x_0, and let $|x-x_0|<\delta$. Then by the mean value theorem, applied to the function $f_m(x)-f_n(x)$, and the condition that (ii) is uniform at x_0, we have, for $m>N$, $n>N$,

$$\begin{aligned}(14)\quad &\left|\frac{f_m(x)-f_m(x_0)}{x-x_0}-f_m'(x_0)-\frac{f_n(x)-f_n(x_0)}{x-x_0}+f_n'(x_0)\right|\\ &\qquad=\left|f_m'(x')-f_m'(x_0)-f_n'(x')+f_n'(x_0)\right|<\epsilon,\end{aligned}$$

where x' may vary with m and n, but always lies between x_0 and x.

Now by the definition of derivative, for any fixed $n>N$ there exists a positive $\delta'<\delta$ such that for $|x-x_0|<\delta'$ we have

$$(15)\qquad \left|\frac{f_n(x)-f_n(x_0)}{x-x_0}-f_n'(x_0)\right|<\epsilon.$$

Adding (14) and (15), we have, if $|x-x_0|<\delta'$ and $m>N$,

$$(16)\qquad \left|\frac{f_m(x)-f_m(x_0)}{x-x_0}-f_m'(x_0)\right|<2\epsilon.$$

Hence the condition of the lemma is fulfilled, and the conclusion of the theorem follows.

COROLLARY A. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist on* $[a, b]$ *and that the derived sequence* (ii) $\{f_n'(x)\}$ *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that* (ii) *be uniform at every point of* $[a, b]$.

COROLLARY B. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *then a necessary and sufficient condition that* $f'(x)$ *exist and be continuous on* $[a, b]$, *and that the derived sequence* (ii) $\{f_n'(x)\}$ *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that* (ii) *be normal at every point of* $[a, b]$.

COROLLARY C. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the functions of the derived sequence* (ii) $\{f_n'(x)\}$ *are continuous on* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist, and that the sequence* (ii) *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that the derivatives* $f_n'(x)$ *be equicontinuous at every point of* $[a, b]$.

COROLLARY D. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$ *and the second derivatives* $f_n''(x)$ *exist and are uniformly bounded on* $[a, b]$, *then* $f'(x)$ *exists and the derived sequence* (ii) $\{f_n'(x)\}$ *converges uniformly to* $f'(x)$ *on* $[a, b]$.

Corollary D follows from Corollary C, since the first derivatives are equicontinuous if the second derivatives are uniformly bounded. The others follow from Theorems 1 and 3, the relations mentioned above between the different conditions, and the properties of equicontinuous and normal sequences.

THEOREM 4. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist and are summable on* $[a, b]$, *and the derived sequence* (ii) $\{f_n'(x)\}$ *converges in the mean on* $[a, b]$, *that is,* $\int_a^b |f_m'(x) - f_n'(x)| dx \to 0$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist and be continuous on* $[a, b]$ *and that the sequence* (ii) *converge to* $f'(x)$ *on* $[a, b]$ *is that* (ii) *be almost normal at every point of* $[a, b]$.

If (ii) is almost normal at x_0, given $\epsilon > 0$ select $\delta > 0$ in accordance with the definition of almost normal, and let $|x - x_0| < \delta$. Let E_k denote the set of values of t between x_0 and x such that for some $n > k$,

$$|f_n'(t) - f_n'(x_0)| \geqq \epsilon, \tag{17}$$

and let C_k be the complement of E_k with respect to the interval $[x_0, x]$. Since $|t - x_0| < \delta$, and (ii) is almost normal at x_0, no value of t is a member of more than a finite number of the sets E_k. Hence the measure of E_k approaches zero. Now

$$\begin{aligned} \left| \int_{x_0}^{x} [f_n'(t) - f_n'(x_0)] dt \right| &\leqq \int_{C_k} |f_n'(t) - f_n'(x_0)| \, dt \\ &+ \int_{E_k} |f_n'(t)| \, dt + \int_{E_k} |f_n'(x_0)| \, dt. \end{aligned} \tag{18}$$

If $n>k$, the first integral on the right of (18) is less than $\epsilon|x-x_0|$, since on C_k the integrand is $<\epsilon$. By the convergence in the mean, we may select m such that, for $n>m$,

$$(19)\qquad \int_a^b |f_m'(t) - f_n'(t)|\,dt < \frac{\epsilon}{2}|x - x_0|.$$

Since the measure of E_k approaches zero, $k>m$ can be selected such that

$$(20)\qquad \int_{E_k} |f_m'(t)|\,dt < \frac{\epsilon}{2}|x - x_0|,$$

since m is now fixed, and the integral in (20) is absolutely continuous. Hence from (19) and (20), for $n>k$,

$$(21)\qquad \int_{E_k} |f_n'(t)|\,dt < \epsilon|x - x_0|.$$

But this is the second integral on the right of (18). Now the sequence $\{f_n'(x_0)\}$ is bounded, for if it were not, there would exist a subsequence of (ii) which diverged to infinity at every point of some neighborhood of x_0, since (ii) is almost normal at x_0. But this is impossible, since (ii) converges in the mean, and every subsequence contains in turn a subsequence which converges almost everywhere. Because of the boundedness, a k exists such that for $n>k$ the third integral on the right of (18) is less than $\epsilon|x-x_0|$. Combining these results, we see that a k exists such that for $n>k$,

$$(22)\qquad \left|\int_{x_0}^{x} [f_n'(t) - f_n'(x_0)]dt\right| < 3\epsilon|x - x_0|.$$

Since $f_n'(x)$ is summable, there results on dividing (22) by $|x-x_0|$, for $n>k$,

$$(23)\qquad \left|\frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0)\right| < 3\epsilon.$$

This is the condition of the lemma; therefore $f_n'(x)$ exists, and (ii) converges to it at x_0, and hence at every point of $[a, b]$. By Theorem 1, $f'(x)$ is continuous on $[a, b]$. Conversely, if $f'(x)$ is continuous on $[a, b]$ and (ii) converges to it, by Theorem 1 the sequence (ii) is almost normal at every point of $[a, b]$.

THEOREM 5. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist and are summable on* $[a, b]$, *and the derived sequence* (ii) $\{f_n'(x)\}$ *converges in the mean on* $[a, b]$, *then a necessary and sufficient condition that the sequence* (ii) *converge on* $[a, b]$ *to some function* $g(x)$ (*not necessarily equal to* $f'(x)$) *is that* (ii) *be almost uniform at every point of* $[a, b]$.

If (ii) is almost uniform at x_0, given $\epsilon > 0$, let us select $\delta > 0$ in accordance with the definition of almost uniform, and let $|x - x_0| < \delta$. By the same method of proof that was used in Theorem 4, we may show that a k exists, such that if $m > k$, $n > k$,

$$(24)\quad \left| \int_{x_0}^{x} [f_m'(t) - f_m'(x_0) - f_n'(t) + f_n'(x_0)]dt \right| < 3\epsilon \mid x - x_0 \mid.$$

In fact, since (ii) is almost uniform, the double sequence $\{f_m'(x) - f_n'(x)\}$ is almost normal, and (24) corresponds to formula (22) in the proof of Theorem 4, applied to this double sequence. Dividing (24) by $|x - x_0|$, we have, for $m > k$, $n > k$,

$$(25)\quad \left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - f_m'(x_0) - \frac{f_n(x) - f_n(x_0)}{x - x_0} + f_n'(x_0) \right| < 3\epsilon.$$

Since (i) converges at x and x_0, an $N > k$ exists such that if $m > N$, $n > N$,

$$(26)\quad \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| < \epsilon.$$

Adding (25) and (26), we have, for $m > N$, $n > N$,

$$(27)\quad | f_m'(x_0) - f_n'(x_0) | < 4\epsilon.$$

Therefore (ii) converges at x_0, and hence at every point of $[a, b]$. Conversely, if (ii) converges on $[a, b]$, by Theorem 1 it is almost uniform at every point of $[a, b]$. That the limit function of (ii) need not be $f'(x)$ can be seen from an example. Let (i) be the sequence $\{x^n/n\}$ on the interval $[0, 1]$. Then the derived sequence converges on $[0, 1]$, but not to the derivative of the limit function when $x = 1$.

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