ON THE SINGULARITIES OF AN ANALYTIC FUNCTION*

BY E. W. MILLER

1. *Introduction*. We shall consider an analytic function $f(z)$ represented by the series $\sum_{n=0}^{\infty} a_n z^n$ whose circle of convergence we shall suppose for simplicity to be the *unit* circle with center at the origin of the complex plane. Our purpose is to give simple generalizations of certain theorems of Pringsheim and Hadamard relative to the singularities of $f(z)$ on C, the circumference of the circle of convergence.

With the present hypotheses and notation the theorems in question may be formulated as follows:

THEOREM OF PRINGSHEIM.† *In order that $z=1$ be a simple pole of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_{n+1} - a_n \right|^{1/n} < 1.$$

THEOREM OF HADAMARD.‡ *In order that there be just one simple pole and no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_n^2 - a_{n-1}a_{n+1} \right|^{1/n} < 1.$$

2. *Generalizations of the Above Theorems*. We shall first establish a generalization of Pringsheim's theorem.

THEOREM 1. *In order that $z=1$ be a pole of order m of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that there exist a polynomial $g(x)$ of degree $m-1$ such that, if we put $A_n = a_n/g(n)$, we have*

$$\overline{\lim_{n\to\infty}} \left| A_{n+1} - A_n \right|^{1/n} < 1.$$

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* Presented to the Society, April 20, 1935.

† A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, vol. 2, part 2, 2d ed., 1932, p. 916.

‡ See, for instance, P. Dienes, *The Taylor Series*, p. 333.

PROOF. Suppose that $z=1$ is a pole of order m of $f(z)$ and that there is no further singularity of $f(z)$ on C. Then $f(z)=f_1(z)+f_2(z)$, where $f_1(z)$ has $z=1$ as a pole of order m and has no other singularity in the complex plane, and where $f_2(z)$ has no singularity within or on C. According to a theorem of Leau and Wigert* there is a polynomial $g(x)$ of degree $m-1$ such that $f_1(z)=\sum_{n=0}^{\infty}g(n)z^n$. We choose this as the polynomial $g(x)$ of Theorem 1. We have then $a_n=g(n)+b_n$, where $\overline{\lim}_{n\to\infty}|b_n|^{1/n}<1$. Hence $A_n=1+B_n$, where $B_n=b_n/g(n)$. Clearly $\overline{\lim}_{n\to\infty}|B_n|^{1/n}<1$. It follows at once that $\overline{\lim}_{n\to\infty}|A_{n+1}-A_n|^{1/n}<1$.

To prove the condition sufficient we notice that by the theorem of Pringsheim cited above $\sum_{n=0}^{\infty}A_nz^n$ has $z=1$ as a simple pole and has no further singularity on C. Thus

$$\sum_{n=0}^{\infty} A_nz^n = k\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} c_nz^n, \quad \text{where } \overline{\lim_{n\to\infty}} |c_n|^{1/n} < 1.$$

Therefore

$$\sum_{n=0}^{\infty} a_nz^n = \sum_{n=0}^{\infty} g(n)A_nz^n = k\sum_{n=0}^{\infty} g(n)z^n + \sum_{n=0}^{\infty} g(n)c_nz^n.$$

Now by the Leau-Wigert theorem $\sum_{n=0}^{\infty}g(n)z^n$ has $z=1$ as a pole of order m and has no other singularity in the complex plane. Furthermore $\overline{\lim}_{n\to\infty}|g(n)c_n|^{1/n}<1$. Hence $\sum_{n=0}^{\infty}a_nz^n$ has $z=1$ as a pole of order m and has no further singularity on C.

In case the type of pole is specified it may be easy to determine the polynomial $g(x)$. For example, suppose we have $f(z)=(k/(1-z)^m)+F(z)$, where the Maclaurin series for $F(z)$ has a radius of convergence >1. Since

$$1/(1-z)^m=\frac{1}{(m-1)!}\sum_{n=0}^{\infty}(n+m-1)\cdots(n+1)z^n,$$

we may put $g(n)=(k/(m-1)!)\ (n+m-1)\cdots(n+1)$. After simplification our condition that $f(z)$ be of the above described type assumes the form

$$\overline{\lim_{n\to\infty}} |(n+m)a_n - (n+1)a_{n+1}|^{1/n} < 1.$$

That this condition is not merely necessary but also sufficient follows easily once we put $a_n=(n+m-1)\cdots(n+1)\,k_n$.

* See, for instance, Dienes, loc. cit., pp. 337–339.

The theorem of Hadamard cited in (1) is susceptible of a similar generalization.

THEOREM 2. *In order that there be just one pole of order* m *and no further singularity of* $f(z)$ *on* C *it is necessary and sufficient that there exist a polynomial* $g(x)$ *of degree* $m-1$ *such that if we put* $A_n = a_n/g(n)$, *then*

$$\overline{\lim_{n\to\infty}} \left| A_n^2 - A_{n-1}A_{n+1} \right|^{1/n} < 1.$$

PROOF. Suppose there is a pole of order m and no further singularity of $f(z)$ on C. Let this be the point $z=b$. If we put $z=bw$, the series $\sum_{n=0}^{\infty} a_n b^n w^n$ will have $w=1$ as a pole of order m and will have no other singularity on the circumference of the circle of convergence. Accordingly there exists a polynomial $g(x)$ of degree $m-1$ such that $a_n b^n = g(n) + c_n'$, where $\overline{\lim}_{n\to\infty} |c_n'|^{1/n} <1$. Hence $A_n = (1/b^n) + c_n$, where $\overline{\lim}_{n\to\infty} |c_n|^{1/n} < 1$. The condition of our theorem now follows easily.

That the condition is sufficient follows by an argument similar to that used in the proof of the sufficiency of the condition in Theorem 1.

Again, it is easy to show that $f(z) = (k/(\alpha - z)^m) + F(z)$, where $|\alpha| = 1$ and where the Maclaurin series for $F(z)$ has a radius of convergence >1, if and only if

$$\overline{\lim_{n\to\infty}} \left| n(n+m)a_n^2 - (n+1)(n+m-1)a_{n-1}a_{n+1} \right|^{1/n} < 1.$$

3. *A Theorem on Limits.* The most important step in the proof of Pringsheim's theorem is to show that, under the conditions of the theorem, $\lim_{n\to\infty} a_{n+1}/a_n = 1$. In our generalization of this theorem it may be noticed that we have

$$\overline{\lim_{n\to\infty}} \left| a_{n+1} - \frac{g(n+1)}{g(n)} a_n \right|^{1/n} < 1$$

and that again $\lim_{n\to\infty} a_{n+1}/a_n = 1$. These facts suggest the following theorem on limits.

THEOREM 3. *If* $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1$, *in order that* $\lim_{n\to\infty} a_{n+1}/a_n = 1$, *it is necessary and sufficient that there exist a sequence of complex numbers* λ_n *such that*

$$(1) \quad \lim_{n\to\infty} \lambda_n = 1; \quad \textit{and} \quad (2) \quad \overline{\lim_{n\to\infty}} \left| a_{n+1} - \lambda_n a_n \right|^{1/n} < 1.$$

PROOF. The necessity is obvious. We need merely put $\lambda_n = a_{n+1}/a_n$. To prove the sufficiency we notice first that if we can prove that $\lim_{n\to\infty}|a_n|^{1/n}$ exists, then $\lim_{n\to\infty}a_{n+1}/a_n = 1$. For from (2) it would follow that

$$\overline{\lim_{n\to\infty}} \; |a_n|^{1/n} \left| \frac{a_{n+1}}{a_n} - \lambda_n \right|^{1/n} < 1,$$

and therefore $\lim_{n\to\infty}a_{n+1}/a_n = 1$. We shall accordingly prove that $\lim_{n\to\infty}|a_n|^{1/n} = 1$. There exists a positive number $k<1$ such that for n sufficiently large $|a_{n+1} - \lambda_n a_n| < k^n$, so that

$$|a_{n+1} - a_n| < k^n + |\epsilon_n|\,|a_n|,$$

where $\lim_{n\to\infty}\epsilon_n = 0$,

$$|a_{n+2} - a_{n+1}| < k^{n+1} + |\epsilon_{n+1}|\,|a_{n+1}|,$$

and in general

$$|a_{n+j} - a_{n+j-1}| < k^{n+j-1} + |\epsilon_{n+j-1}|\,|a_{n+j-1}|.$$

Therefore

$$|a_{n+j} - a_n| < \frac{k^n}{1-k} + |\epsilon_n|\,|a_n| + \sum_{r=n+1}^{n+j-1} |\epsilon_r|\,|a_r|.$$

Now let us suppose that $\overline{\lim}_{n\to\infty}|a_n|^{1/n} \neq \underline{\lim}_{n\to\infty}|a_n|^{1/n}$. We can find α and β such that $k<\beta<1$ and $\alpha<\beta$, and two increasing sequences of positive integers $\{p\}$ and $\{q\}$ such that $|a_p| > \beta^p$, $|a_q| < \alpha^q$ and $|a_n| \leqq \beta^n$ if n is not a term of the sequence $\{p\}$. It is obvious that from $\{p\}$ and $\{q\}$ we can select subsequences $\{p_i\}$ and $\{q_i\}$ of such a nature that $p_i < q_i$ and that if $p_i < n < q_i$, then $|a_n| \leqq \beta^n$. We have of course $|a_{p_i}| > \beta^{p_i}$ and $|a_{q_i}| < \alpha^{q_i}$. Accordingly we have $|a_{p_i} - a_{q_i}| > |a_{p_i}| - \alpha^{q_i}$. On the other hand, from our preliminary result we have

$$|a_{p_i} - a_{q_i}| = |a_{q_i} - a_{p_i}| < \frac{k^{p_i}}{1-k} + |\epsilon_{p_i}|\,|a_{p_i}| + \sum_{r=p_i+1}^{q_i-1} |\epsilon_r|\,|a_r|.$$

Therefore

$$|a_{p_i} - a_{q_i}| < \frac{k^{p_i}}{1-k} + |\epsilon_{p_i}|\,|a_{p_i}| + \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta},$$

where ϵ'_{p_i} denotes the largest of the numbers $|\epsilon_r|$, $(r = p_i+1, \cdots, q_i - 1)$. We now consider the difference

$$[\,|a_{p_i}| - \alpha^{q_i}] - \left[\frac{k^{p_i}}{1-k} + |\epsilon_{p_i}|\,|a_{p_i}| + \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta}\right]$$

$$= |a_{p_i}|(1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta}$$

$$> \beta^{p_i}(1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta}$$

$$= \beta^{p_i}\left(1 - |\epsilon_{p_i}| - \frac{\alpha^{q_i}}{\beta^{p_i}} - \frac{k^{p_i}}{\beta^{p_i}(1-k)} - \epsilon'_{p_i}\frac{\beta}{1-\beta}\right).$$

Now for i sufficiently large all the terms within the last parentheses except the first are as small as we please. Hence for sufficiently large i the difference in question is positive. From this contradiction the theorem follows.

In conclusion, we may note as a simple corollary of the above theorem that if $\overline{\lim}_{n\to\infty}|a_n|^{1/n}=1$, then $\lim_{n\to\infty}|a_{n+1}/a_n|=1$ if and only if there exists a sequence of *real* numbers λ_n such that $\lim_{n\to\infty}\lambda_n=1$ and $\overline{\lim}_{n\to\infty}|\,|a_{n+1}|-\lambda_n|a_n|\,|^{1/n}<1$.

THE UNIVERSITY OF MICHIGAN

ON THE COEFFICIENTS OF A TYPICALLY-REAL FUNCTION*

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1. *Introduction.* It is well known‡ that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

is regular for $|z| \leqq 1$, and if E is defined by the formula

$$E \equiv \underset{|z_1|=|z_2|=1}{\text{maximum}} \left|\mathcal{R}f(z_1) - \mathcal{R}f(z_2)\right|, \tag{2}$$

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† National Research Fellow.

‡ See E. Landau, Archiv der Mathematik und Physik, (3), vol. 11 (1906), pp. 31–36.