

$$\begin{aligned}
& [|a_{p_i}| - \alpha^{q_i}] - \left[\frac{k^{p_i}}{1-k} + |\epsilon_{p_i}| |a_{p_i}| + \epsilon'_{p_i} \frac{\beta^{p_i+1}}{1-\beta} \right] \\
&= |a_{p_i}| (1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i} \frac{\beta^{p_i+1}}{1-\beta} \\
&> \beta^{p_i} (1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i} \frac{\beta^{p_i+1}}{1-\beta} \\
&= \beta^{p_i} \left(1 - |\epsilon_{p_i}| - \frac{\alpha^{q_i}}{\beta^{p_i}} - \frac{k^{p_i}}{\beta^{p_i}(1-k)} - \epsilon'_{p_i} \frac{\beta}{1-\beta} \right).
\end{aligned}$$

Now for i sufficiently large all the terms within the last parentheses except the first are as small as we please. Hence for sufficiently large i the difference in question is positive. From this contradiction the theorem follows.

In conclusion, we may note as a simple corollary of the above theorem that if $\overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} = 1$, then $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$ if and only if there exists a sequence of real numbers λ_n such that $\lim_{n \rightarrow \infty} \lambda_n = 1$ and $\overline{\lim}_{n \rightarrow \infty} | |a_{n+1}| - \lambda_n |a_n| |^{1/n} < 1$.

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ON THE COEFFICIENTS OF A TYPICALLY- REAL FUNCTION*

BY M. S. ROBERTSON†

1. *Introduction.* It is well known‡ that if

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is regular for $|z| \leq 1$, and if E is defined by the formula

$$(2) \quad E \equiv \text{maximum}_{|z_1|=|z_2|=1} | \mathcal{R}f(z_1) - \mathcal{R}f(z_2) |,$$

* Presented to the Society, February 23, 1935.

† National Research Fellow.

‡ See E. Landau, *Archiv der Mathematik und Physik*, (3), vol. 11 (1906), pp. 31-36.

or, in other words, if E is the oscillation of the real part of $f(z)$ for all points z_1 and z_2 on the unit circle, then

$$(3) \quad |a_1| = |f'(0)| \leq \frac{2}{\pi} E.$$

In this paper an analogous result and extensions are obtained for *all* the coefficients of any function $f(z)$ regular and *typically-real* for $|z| < 1$.

DEFINITION. A function $f(z)$, $f(0) = 0$, $f'(0) \neq 0$, regular for $|z| < R$, is said to be typically-real with respect to the circle $|z| = R$, if within this circle $f(z)$ is real for, and only for, the points on the real axis.*

It may be noticed, as W. Rogosinski has pointed out, that the class of functions regular and univalent in the circle $|z| = R$ and real on the real axis form a subclass of the class of functions typically-real with respect to this circle.

2. *A Stieltjes Integral Representation for Typically-Real Functions.* Let

$$(4) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \text{ real}),$$

be regular and typically-real for $|z| < 1$. Then it is known† that $f(z)$ can be represented in the form

$$(5) \quad f(z) = \frac{zg(z)}{1 - z^2},$$

where $g(z)$ is regular for $|z| < 1$, $g(0) = 1$, $\Re g(z) > 0$ for $|z| < 1$. Further, by the formula of G. Herglotz,‡ we may write

$$(6) \quad \begin{aligned} g(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} d\alpha(\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - z^2 - 2iz \sin \theta}{1 - 2z \cos \theta + z^2} d\alpha(\theta), \end{aligned}$$

* See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, *Mathematische Zeitschrift*, vol. 35 (1932), pp. 93–121.

† See W. Rogosinski, *loc. cit.*, p. 99.

‡ See G. Herglotz, *Leipziger Berichte*, 1911, pp. 501–511.

where $\alpha(\theta)$ is an odd non-decreasing function of θ in the interval $(-\pi, \pi)$, and where $f(z)$ is real on the real axis, as is also $g(z)$. Hence

$$(7) \quad \begin{aligned} f(z) &= \frac{1}{\pi} \int_0^\pi \frac{z d\alpha(\theta)}{1 - 2z \cos \theta + z^2} \\ &= \frac{1}{\pi} \int_0^\pi \left(\sum_{n=1}^{\infty} \frac{\sin n\theta}{\sin \theta} z^n \right) d\alpha(\theta). \end{aligned}$$

L. Fejér has observed* that

$$(8) \quad F(z) = \int_0^z \frac{f(z)}{z} dz = z + \sum_2^{\infty} \frac{a_n}{n} z^n,$$

is univalent and convex in the direction of the imaginary axis for $|z| < 1$, that is, no straight line parallel to the imaginary axis can cut the image of the circle $|z| = r$ (for every r in the interval $0 < r < 1$) mapped by the function $F(z)$ in more than two points. It follows from (7) and (8) by integration that

$$(9) \quad F(z) = \frac{1}{2\pi i} \int_0^\pi \log \left\{ \frac{1 - ze^{-i\theta}}{1 - ze^{i\theta}} \right\} \frac{d\alpha(\theta)}{\sin \theta}.$$

3. *The Coefficients of a Typically-Real Function.* From (9), since $F(r)$ is real, we have

$$(10) \quad F(r) = \frac{1}{\pi} \int_0^\pi \arg(1 - re^{-i\theta}) \frac{d\alpha(\theta)}{\sin \theta}.$$

Since the integrand is an increasing function of r for every θ , we have

$$(11) \quad \begin{aligned} F(1) &\equiv \lim_{r \rightarrow 1} F(r) = \frac{1}{\pi} \int_0^\pi \lim_{r \rightarrow 1} \arg(1 - re^{-i\theta}) \frac{d\alpha(\theta)}{\sin \theta} \\ &= \frac{1}{2\pi} \int_0^\pi \frac{\pi - \theta}{\sin \theta} d\alpha(\theta). \end{aligned}$$

Similarly we also have

$$(12) \quad F(-1) \equiv \lim_{r \rightarrow 1} F(-r) = \frac{-1}{2\pi} \int_0^\pi \frac{\theta}{\sin \theta} d\alpha(\theta).$$

* See L. Fejér, *Journal of the London Mathematical Society*, vol. 8 (1933), p. 61, footnote.

These limits are finite or infinite according as the integrals exist or do not exist. Hence if we define

$$(13) \quad E(r) \equiv F(r) - F(-r) = \int_{-r}^r \frac{f(t)}{t} dt = \int_{-1}^{+1} \frac{f(rt)}{t} dt,$$

then

$$(14) \quad E \equiv \lim_{r \rightarrow 1} E(r) = \frac{1}{2} \int_0^\pi \frac{d\alpha(\theta)}{\sin \theta},$$

and is finite whenever this integral exists. However, since $F(z)$ is convex in the direction of the imaginary axis, and since $E(r)$ is the length of the segment of the real axis intercepted by the contour into which $|z| = r$ is mapped by $F(z)$, we have

$$(15) \quad |\Re F(z_1) - \Re F(z_2)| \leq F(r) - F(-r) = E(r)$$

for all z_1 and z_2 on $|z| = r$. Thus $E(r)$ denotes the oscillation of the real part of $F(z)$ on $|z| = r$.

From (4) and (7) we have, by comparing coefficients on both sides of the equation (7),

$$(16) \quad a_n = \frac{1}{\pi} \int_0^\pi \frac{\sin n\theta}{\sin \theta} d\alpha(\theta),$$

$$(17) \quad |a_n| \leq \frac{1}{\pi} \int_0^\pi \left| \frac{\sin n\theta}{\sin \theta} \right| d\alpha(\theta) \leq \frac{1}{\pi} \int_0^\pi \frac{d\alpha(\theta)}{\sin \theta}.$$

Whenever E is finite we have, by (14) and (17),

$$(18) \quad |a_n| \leq \frac{2}{\pi} E, \quad \text{for all } n,$$

$$(19) \quad \frac{1}{n+1} \sum_{k=1}^n |a_k| \leq \frac{1}{\pi} \int_0^\pi M_n(\theta) \frac{d\alpha(\theta)}{\sin \theta},$$

where

$$M_n(\theta) \equiv \frac{1}{n+1} \sum_{k=1}^n |\sin k\theta|.$$

However, as T. Gronwall has shown,*

* See T. Gronwall, Transactions of this Society, vol. 13 (1912), pp. 445-468.

$$(20) \quad M_n(\theta) < \sin z_0 = 0.72457 \dots,$$

where z_0 is the positive root of the equation $\tan(z_0/2) = z_0$.
Further,

$$(21) \quad \lim_{n \rightarrow \infty} M_n(\theta) = M(\theta) \leq \frac{2}{\pi},$$

$$(22) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |a_k| \leq \frac{1}{\pi} \int_0^\pi \lim_{n \rightarrow \infty} M_n(\theta) \frac{d\alpha(\theta)}{\sin \theta}$$

$$\leq \frac{2}{\pi} \frac{1}{\pi} \int_0^\pi \frac{d\alpha(\theta)}{\sin \theta} \leq \left(\frac{2}{\pi}\right)^2 E.$$

Similarly,

$$(23) \quad \frac{1}{n+1} \sum_{k=1}^n |a_k| < \left(\frac{2 \sin z_0}{\pi}\right) E < \left(\frac{1.45}{\pi}\right) E,$$

for all n .

Again, if we denote by Γ_n the expression

$$(24) \quad \Gamma_n = \max_{\theta} \sum_{k=1}^n \frac{|\sin k\theta|}{k},$$

then we have*

$$(25) \quad \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} < \Gamma_n < \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{2}{\pi},$$

$$(26) \quad \lim_{n \rightarrow \infty} \frac{\Gamma_n}{\log n} = \frac{2}{\pi}.$$

Hence by the method used above in (19) we may show that

$$(27) \quad \sum_{k=1}^n \frac{|a_k|}{k} < \left\{ \left(\frac{2}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2 \sum_{k=1}^n \frac{1}{k} \right\} E,$$

$$(28) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{\log n} \cdot \sum_{k=1}^n \frac{|a_k|}{k} \leq \left(\frac{2}{\pi}\right)^2 E.$$

Let

* See G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, 1925, pp. 81 and 274.

$$(29) \quad A(n, \theta) \equiv \sum_{k=1}^n \frac{\sin k\theta}{k}.$$

Then the absolute maximum* of $A(n, \theta)$ is $A(n, \pi/(n+1))$. Consequently we obtain by the above method

$$(30) \quad \left| \sum_{k=1}^n \frac{a_k}{k} \right| \leq \frac{2}{\pi} A\left(n, \frac{\pi}{n+1}\right)E.$$

For n odd the factor $2/\pi$ in (18) cannot be replaced by a smaller one, since for the function $f(z) = z(1+z^2)^{-1}$ we have

$$|a_{2n-1}| = 1, \quad a_{2n} = 0, \quad F(z) = \arctan z, \quad E = \frac{\pi}{2}.$$

Hence equality is attained by $z(1+z^2)^{-1}$ for every odd value of n . However, one cannot have equality for all n , even and odd, for a given function of the class under consideration, as this would contradict the inequality (22).

4. *A Class of Odd Typically-Real Functions.* Let I denote the class of odd functions

$$(31) \quad f(z) = z + \sum_{n=1}^{\infty} b_{2n+1}z^{2n+1}$$

with the properties

- (a) $f(z)$ is regular for $|z| < 1$,
- (b) $f(z)$ is real on the real axis, that is, b_{2n+1} is real for all n ,
- (c) $f(z)$ lies inside the j th quadrant whenever z is inside the j th quadrant for $|z| < 1$, ($j = 1, 2, 3, 4$).

The class of odd functions regular and univalent for $|z| < 1$ and real on the real axis form a subclass of I .

THEOREM. *If*

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1}z^{2n+1}$$

belongs to class I, then

$$|b_3| \leq 1, \quad |b_{2n-1}| + |b_{2n+1}| \leq 2, \quad \sum_{k=1}^n \frac{b_{2k+1}}{k} \geq -2.$$

* See G. Pólya and G. Szegő, loc. cit., p. 79.

PROOF. Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

$$(32) \quad b_{2n+1}r^{2n+1} = \frac{1}{\pi} \int_{-\pi}^{\pi} v(r, \theta) \sin (2n + 1)\theta \, d\theta,$$

$$(33) \quad b_{2n+1}r^{2n+1} = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos (2n + 1)\theta \, d\theta.$$

On account of hypotheses (b) and (c) of the definition of class I we may write

$$(34) \quad b_{2n+1}r^{2n+1} = \frac{4}{\pi} \int_0^{\pi/2} v(r, \theta) \sin (2n + 1)\theta \, d\theta,$$

$$(35) \quad b_{2n+1}r^{2n+1} = \frac{4}{\pi} \int_0^{\pi/2} u(r, \theta) \cos (2n + 1)\theta \, d\theta,$$

where $v(r, \theta) > 0$, $u(r, \theta) > 0$ for $0 < \theta < \pi/2$, $r < 1$. From (34) we have

$$(36) \quad b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1} = \frac{8}{\pi} \int_0^{\pi/2} v(r, \theta) \cos 2n\theta \sin \theta \, d\theta,$$

so that

$$\begin{aligned} |b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1}| &\leq \frac{8}{\pi} \int_0^{\pi/2} |v(r, \theta) \sin \theta| \, d\theta \\ &\leq \frac{8}{\pi} \int_0^{\pi/2} v(r, \theta) \sin \theta \, d\theta \\ &\leq 2r. \end{aligned}$$

Letting $r \rightarrow 1$, we have

$$(37) \quad |b_{2n-1} - b_{2n+1}| \leq 2.$$

From (35) we have similarly

$$\begin{aligned} |b_{2n-1}r^{2n-1} + b_{2n+1}r^{2n+1}| &\leq \frac{8}{\pi} \int_0^{\pi/2} |u(r, \theta) \cos 2n\theta \cos \theta| \, d\theta \\ (38) \quad &\leq \frac{8}{\pi} \int_0^{\pi/2} u(r, \theta) \cos \theta \, d\theta \\ &\leq 2r. \end{aligned}$$

Letting $r \rightarrow 1$ again, we have

$$(39) \quad |b_{2n-1} + b_{2n+1}| \leq 2.$$

Combining (37) and (39) we obtain

$$(40) \quad |b_{2n-1}| + |b_{2n+1}| \leq 2, \quad (\text{for all } n).$$

The inequalities (40) were established by a different method for the subclass of I consisting of odd univalent functions real on the real axis by J. Dieudonné.* Further, since we have†

$$(41) \quad B(n, \theta) \equiv \sum_{k=1}^n \frac{\cos k\theta}{k} \geq -1$$

for all n , then

$$(42) \quad \sum_{k=1}^n \frac{(b_{2k+1}r^{2k+1} - b_{2k-1}r^{2k-1})}{k} \\ = \frac{8}{\pi} \int_0^{\pi/2} B(n, \theta)v(r, \theta) \sin \theta \, d\theta \geq -2r,$$

$$(43) \quad \sum_{k=1}^n \frac{(b_{2k+1} - b_{2k-1})}{k} \geq -2,$$

and similarly

$$(44) \quad \sum_{k=1}^n \frac{(b_{2k+1} + b_{2k-1})}{k} \geq -2.$$

On adding (43) and (44) we obtain also

$$(45) \quad \sum_{k=1}^n \frac{b_{2k+1}}{k} \geq -2.$$

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* See J. Dieudonné, *Annales de l'École Normale*, vol. 48 (1931), p. 318.

† See G. Pólya und G. Szegő, *loc. cit.*, p. 79.