QUINE ON LOGISTIC

A System of Logistic. By Willard Van Orman Quine. Harvard University Press, 1934. x+204 pp.

In this book is presented a system of symbolic logic based on that of Whitehead and Russell's *Principia Mathematica*, but involving a number of fundamental changes. The most important of these changes are: (1) the representation of functions of two or more variables as functions of one variable through the introduction, as an undefined term, of the operation of ordination, that is, the operation of combining two elements a and b into the ordered pair a,b; (2) the use of this same notion of ordination to replace the notion of predication, the proposition ϕa, obtained by predicating the propositional function ϕ of the argument a, being identified with the ordered pair ϕ,a; (3) the introduction in connection with the operation of abstraction, ^, of a rule of inference, the rule of concretion, which takes the place of that tacit rule of *Principia* which, to speak somewhat inexactly, allows the substitution for ϕx, in any proved expression in which ϕ is a free variable, of any appropriate expression containing x; (4) a liberalization of the theory of types, by which the axiom of reducibility is rendered unnecessary; (5) the use of the notion of classial referent, introduced by an actual nominal definition, to replace almost entirely the clumsy descriptions introduced in *Principia* as incomplete symbols; (6) the introduction, under the name of congeneration, of the relation of implication between propositional functions, as an undefined term, out of which both the relation of implication between propositions and the universal and existential quantifiers are obtained by definition.

Quine's propositional functions have the property that equivalence implies equality, and for this reason he speaks of them as classes rather than as propositional functions. Nevertheless he uses them for the purposes for which propositional functions are used in *Principia* and in other systems, and hence, for the sake of comparison, we continue to call them propositional functions.

In regard to Quine's use of ordination, it is, of course, clear, as he points out, that the introduction as primitive ideas of an infinite number of different notions of predication, one for functions of one variable, another for functions of two variables, another for functions of three variables, and so on, is awkward and that it is therefore desirable to find some device by which functions of two or more variables can be regarded as special cases of functions of one variable. It is not so clear, however, that the introduction of the ordered pair as an undefined term is the best method of doing this. From some points of view the more natural and more elegant method is that of Schönfinkel,* under which a function of $n+1$ variables is regarded as a function of one variable whose values are functions of n variables. For example, instead of what is ordinarily written $\phi(a, b)$, Schönfinkel writes $(\phi a)b$, where ϕa is regarded as a function which, when taken of the argument b, yields the proposition $(\phi a)b$, and ϕ is regarded as a function which, when taken of the argument a, yields the func-

* Mathematische Annalen, vol. 92 (1924), pp. 305–316.

tion ϕa. Of course, if the Schönfinkel device be adopted, it is necessary to introduce the notion of predication for functions of one variable as a primitive idea, but the number of primitive ideas is not thereby increased, because the ordered pair a,b can then be defined, in terms of predication and abstraction, as $\hat{\phi}((\phi a)b)$, that is, as the class of relations which hold between a and b.

On the other hand, Quine's identification of the notion of predication with that of ordination not only raises the difficult philosophical problem of justifying the assumption that, if x is of one type higher than y, then any assertion about x and y can be construed as an assertion about the proposition x,y, but also introduces unnecessary formal complications, for example, in the rule of concretion. The use of predication as a primitive idea has the advantage that it neither accepts nor denies Quine's special philosophy concerning the nature of predication.

The use of the device of Schönfinkel just referred to is not incompatible with the theory of types, but it does require modification of the usage of *Principia* by which propositional functions are regarded as entities of an entirely different sort from other functions, modification at least to the extent of allowing that functions of one variable whose values are propositions, and functions of one variable whose values are propositional functions, are concepts sufficiently similar so that one notion of application, or predication, and one method of symbolizing this notion, are sufficient for both.

As a matter of fact, it is the contention of the present reviewer that the distinction between propositional functions and functions of other sorts is no more fundamental than, say, the distinction between functions of a real variable and functions of a complex variable, and that the one notion of application, or predication, should suffice for all functions of one variable. Quine, however, following *Principia*, has two notations for the application of a function, one for propositional functions and one for descriptive functions, and in the same way two notations for the operation of abstraction. Thus if M is an expression which contains x as a free variable and which takes on propositions as values when x takes on particular values, he uses $\hat{x}\mathrm{M}$ to denote the corresponding propositional function, and $\hat{x}\mathrm{M},a$ to denote the result of application of this propositional function to the argument a. But if M takes on classes as values when x takes on particular values, then he uses

$$w(\exists,\hat{\alpha}(\exists,x(w = \alpha,x \cdot \alpha = \mathrm{M})))$$

to denote the corresponding function (his usage amounts to that), and $\hat{w}(\exists,\hat{\alpha}(\exists,\hat{x}(w=\alpha,x\cdot\alpha=\mathrm{M})))\text{‘}a$ to denote the result of application of this function to the argument a.

The rule of inference of *Principia* referred to under (3) in the first paragraph above is (like the simple rule of substitution) entirely suppressed by the authors of that work, who use it repeatedly but make no mention of it. Hilbert and Ackermann* make this rule explicit, but their statement of it is inadequate. Quine's revised statement of the rule, on page 187 of his book, is perhaps adequate as applied to the system of *Principia*, but his statement of

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* *Grundzüge der Theoretischen Logik*, p. 53.

what the analogous rule would have to be for his own system is again inadequate, as can be seen, for example, by considering the proposition

$$\alpha,\hat{v}(\alpha,\iota\text{‘}v) \cdot \supset :\cdot \sim \cdot \alpha,\wedge : \supset \cdot \exists,\hat{w}(\alpha,\iota\text{‘}w)$$

and raising the question of substituting for the free variable α on the basis that α,x shall mean $x,\hat{y}(x,[y])$. The following is proposed as a correct statement of what this rule of inference should be for use in Quine's system (in order to make possible in Quine's system the equivalent of what Quine's (26) on page 187 makes possible in the system of *Principia*).

Let A be a significant expression, and let α be a variable whose occurrences as a free variable in A are occurrences as the first symbol of parts of A of the form α,U. We may list these parts as $\alpha,U_1, \cdots, \alpha,U_n$, where $U_1, \cdots, U_n$ are significant, and where the listing is in such an order that, if U_j contains α,U_i, then $i<j$. Let M be a significant expression in which x occurs as a free variable, and let M_1 stand for the result of substituting (in the sense of Quine, page 42) U_1 for x in M. Let A_1 be the expression obtained from A by substituting M_1 for the part α,U_1 and let $\alpha,U_{12}, \cdots, \alpha,U_{1n}$ be the parts of A_1 into which the parts $\alpha,U_2, \cdots, \alpha,U_n$ of A are transformed by the substitution. Let M_2 stand for the result of substituting U_{12} for x in M. Let A_2 be the expression obtained from A_1 by substituting M_2 for the part α,U_{12}, and let $\alpha,U_{23}, \cdots, \alpha,U_{2n}$ be the parts of A_2 into which the parts $\alpha,U_{13}, \cdots, \alpha,U_{1n}$ of A_1 are transformed by the substitution, and so on, until M_n and A_n are defined. If A is a proved expression, the rule which we are stating allows us to infer A_n, provided that A_n is a propositional expression (as defined below).*

The superior simplicity of the rule of concretion, and the advantage of avoiding the foregoing complicated rule by introducing the simpler one (as Quine does), is obvious. In fact, the effect is to analyze a complicated inference into a series of simpler inferences by substitution and concretion, as can be illustrated in connection with the example just mentioned, by substituting $\hat{x}(x,\hat{y}(x,[y]))$ for α in the proposition in question and then making a number of successive applications of the rule of concretion (four or five according to the order in which they are made). Unfortunately, as already remarked, the essential simplicity of the rule of concretion is partially obscured by the peculiar use of ordination as a substitute for predication.

Nowhere in Quine's book is there a definition of the word *significant*, which is used in his statement of the rule of substitution, but from scattered remarks about types and by observation of how the rule of substitution is actually used, it is possible to surmise what probably is meant by the word. Since this is a matter of some importance, especially in view of the fact that it is only through this word (or the related term *propositional expression*) that the theory of types enters the formal system at all, an explicit definition of *significant* is attempted here.

The four metamathematical (or "prosystematic") terms, *significant, classial expression, propositional expression, type*, must be defined simultaneously by induction, as follows. A variable standing alone is significant and may be as-

* With appropriate modifications to adapt it to the notation of Quine, the statement of this rule is taken from a set of notes by S. C. Kleene on lectures of Kurt Gödel, which the reviewer has before him.

signed any type out of the scheme of types explained in Quine's second chapter; if assigned a type of the form $a!$, the variable is a classial expression, and if assigned a type of the form $a! \uparrow a$, it is a propositional expression. If M and N are significant and are assigned the types m and n, respectively, and if it is true of every variable x which occurs in both M and N that the same type was assigned to x in assigning the type m to M that was assigned to x in assigning the type n to N, then (M, N) is significant and must be assigned the type $m \uparrow n$; moreover, if m is $n!$, then (M,N) is a propositional expression. If M is assigned the type m and is a classial expression, then [M] is significant, is a classial expression, and must be assigned the type $m!$. If M is assigned the type m and is a propositional expression, and x is any variable, then $\hat{x}$M is significant, is a classial expression, *and must be assigned the type r! where r is the type that was assigned to x in assigning the type m to* M, *or, if x does not occur in* M, *where r is any type whatever.* When no particular assignment of types to the parts of an expression is in question, the expression shall be called significant (a classial expression, a propositional expression) if types can be assigned to its parts so as to make it significant (a classial expression, a propositional expression).

Quine further requires that an expression set down as a postulate or theorem shall not be considered significant unless it is a propositional expression. But this seems to be an unnecessary complication of terminology, which could be avoided by no greater change than replacing the word "significant" by "a propositional expression" in the statement of the rule of substitution.

The italicized clause in the foregoing definition marks a sharp divergence of Quine's theory of types from that of *Principia Mathematica;* for if the analogy with the theory of types of *Principia* were preserved, $\hat{x}$M could not be of lower type than M.* It is true, of course, in *Principia*, that if α is a class then a proposition of the form $x \epsilon \alpha$ must be of type just one higher than the type of x, but it is to be remembered that this situation is brought about only with the aid of the axiom of reducibility, and that, in any case, the classes of *Principia* are incomplete symbols defined only contextually. Since Quine's $\hat{x}$M is not an incomplete symbol, a truer comparison of the two systems appears to be obtained if we compare expressions in Quine of the form $\hat{x}$M with the propositional functions of *Principia* rather than with the classes of *Principia.* And from this point of view it is seen that, without claiming to do so, Quine has really made an important modification in the theory of types, in a direction which seems to have been first suggested by F. P. Ramsey.†

This modification in the theory of types renders the axiom of reducibility unnecessary in the system of Quine. In particular, the difficulty in regard to the least upper bound of a bounded set of real numbers‡ disappears. For let real numbers be segments of rational numbers, and let λ be a bounded set of real

* See *Principia Mathematica*, 2d ed., vol. 1, p. 48 et seq., and introduction to the second edition, p. xxxix.

† Proceedings of the London Mathematical Society, (2), vol. 25 (1926), p. 362 et seq.

‡ See *Principia Mathematica*, introduction to the second edition, pp. xliv–xlv.

numbers. Then $\epsilon`\lambda$ (replacing the $s`\lambda$ of *Principia*) is the least upper bound of λ, and is of the same type as the real numbers in the set λ.

Quine's statements of his rules of inference require a number of corrections. In his definition, in connection with the rule of substitution, of what he means by the process of substitution, it should be provided that when the bound variable is rewritten in E′ it should be made alphabetically distinct, not only from all variables in E, but also from all variables previously occurring in E′. In the rule of subsumption the proviso should be added that $[\alpha]$, $\hat{x}$(- - -) be a propositional expression; otherwise, since $x,\alpha \cdot \supset \cdot x,\alpha$ is provable, the rule could be used to infer $[\alpha],x(x,\alpha \cdot \supset \cdot x,\alpha)$. In the rule of concretion it should be stipulated that - - - and $\cdots$ be significant; otherwise the rule could be applied to $\hat{z}(\hat{x}(x,y),z),\hat{t}(\exists,\hat{u}(u,t))$, taking - - - to be $(x,y),z$, and $\cdots$ to be $\hat{t}(\exists, \hat{u}(u,t))$.

Apparently a rule of inference allowing an alphabetical change of a bound variable in any theorem or postulate should be added to Quine's four rules, since such a rule of inference is used in the proof of 3.8 on page 83. Of course, it may be that such a rule of inference is unnecessary on the ground that the effect of it can be obtained by some succession of applications of the four rules of inference and the postulates. But if so, this should be explained.

The contention on page 51 that the use of z,x,y to replace $z,(x,y)$ is not to be construed as a definition, or abbreviation, is definitely untenable. For if z,x,y is to be regarded otherwise than as an abbreviated notation for $z,(x,y)$, the system of Quine is open to the same charge as that which he brings against the system of *Principia*, namely, that of being incompletely formalized and leaving lacunae to be bridged by the common sense of the reader. If z,x,y is not an abbreviation, and if the rule of substitution is to be taken literally, then we may substitute z,x for t in the proposition $t,y=u, \hat{p}(\sim p),(\sim \cdot v=v) : \supset y$ and so obtain $z,x,y=u, \hat{p}(\sim p),(\sim \cdot v=v) : \supset y$. In the opinion of the reviewer the remedy for this situation is to introduce the notation for an ordered pair as (x,y) instead of x,y, then to use x,y and z,x,y as abbreviations for (x,y) and $(z,(x,y))$, respectively, whenever no ambiguity is thereby created, of course with the understanding that the rules of inference are applicable only to the unabbreviated form of an expression. In this way the parentheses in (x, y) would become as much a part of the formal system as the brackets in $[\alpha]$, and the notion of parentheses as an extra-formal convention would disappear.

The classial referent of x with respect to the relation α, denoted by $\alpha`x$, is defined in such a way that its intuitive meaning is, "the class of all members of classes bearing the relation $\alpha$ to $x$." Thus, if there is one and only one class bearing the relation $\alpha$ to $x$, then $\alpha`x$ denotes that class. Consequently the classial referent can be used as a description in any case where the thing described is a class, and it happens in the system of Quine that nearly everything worth describing is a class. The superiority of a nominal definition over the use of descriptions as incomplete symbols (as in *Principia*) requires no elaboration.

A considerable economy in the number of primitive ideas is effected by introducing the notion of congeneration, denoted by [ ], as a primitive idea. This notion is explained by Quine on the basis that $[\alpha]$ means the class of classes containing α. But it may also be thought of as implication between propositional functions, because, if α and β are propositional functions (classes), then

$[\alpha],\beta$ is the proposition, "α implies β," expressed in *Principia* as $\alpha x \supset_x \beta x$. It is perhaps worth while to observe that [ ] is a propositional function of two variables, not in the sense of Quine, but in the sense of Schönfinkel, since, if α is a propositional function of one variable, $[\alpha]$ is a propositional function of one variable.

There is no slur on the invaluable pioneer work of Whitehead and Russell when it is said that their system is unsatisfactory from the viewpoints of formal definiteness and of mathematical elegance. The work of Quine is in both respects an important improvement over the system of *Principia*, and, although open to criticism in certain directions, is probably not too highly praised by Whitehead when he calls it, "A landmark in the history of the subject".

ALONZO CHURCH

AMERICAN MATHEMATICS BEFORE 1900

A History of Mathematics in America before 1900. By David Eugene Smith and Jekuthiel Ginsburg. (The Carus Monographs, No. 5.) Mathematical Association of America, 1934. x+209 pp.

The Committee on the Carus Monographs had a happy inspiration when it was led to induce Professor Smith to prepare this history. He was in every way qualified for the task—through his unique knowledge of the subject, through his attractive literary style, and through the excellence of his judgment in dealing with a great mass of material and in presenting its essence in well-balanced and compact form. All of these qualities are very much in evidence in the little volume under review. Only one who has had considerable experience in such matters can truly appreciate the great amount of research which went into the preparation of the manuscript. In this research Professor Smith had the valuable assistance of Professor Ginsburg of Yeshiva College, the editor-in-chief of Scripta Mathematica.

For the purposes of the history "America" was roughly considered as the territory north of the Caribbean Sea and the Rio Grande River. In 1938 fifty years of activity of the American Mathematical Society will be celebrated, and a number of scholars will doubtless cooperate in presenting a historical picture of each of the fields of American mathematics during that period. Such a survey, and the complementary work under review, will thus give an up-to-date panorama of outstanding mathematical activities of the past. The importance of these activities after 1875 for the extraordinary development in the twentieth century will be assessed, and Professor Smith's delineation of milestones of earlier progress will be recalled.

In the sixteenth and seventeenth centuries the mathematical needs of the early American settlers were few, and even at Harvard and William and Mary Colleges, nothing noteworthy was done. Astronomical observations were made to a certain extent, and almanacs prepared; astrologers were by no means unknown. "The century that saw the work of Galileo, Kepler, . . . ,* Napier,

* The name "Gilbert" occurred here in the original sentence (p. 13). The reviewer is unequal to guessing to whom it was intended to refer.