A NOTE ON LIPSCHITZ CLASSES

BY E. S. QUADE

This note consists in the application of some results of Hardy and Littlewood* on fractional integrals to a theorem of Paley and Zygmund† and gives a generalization of that theorem.

We consider only functions of the Fourier power series type. That is, \( f(x) \) is periodic in \( 2\pi \), integrable, and with a Fourier series of the form

\[
f(x) = \sum_{n=0}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.
\]

In dealing with functions of the class \( \text{Lip}(\alpha) \) or \( \text{Lip}(\alpha, p), \alpha \neq 1 \), this restriction is a matter of convenience rather than one of necessity.‡

A function \( f(x) \) is said to belong to the class \( \text{Lip}(\alpha) \), where \( 0 \leq \alpha \leq 1 \), in the interval \( (-\pi, \pi) \), if

\[
f(x + h) - f(x - h) = O(h^\alpha)
\]

uniformly for \( -\pi \leq x - h < x + h \leq \pi \), and to \( \text{Lip}(\alpha, p) \), where \( p \geq 1, 0 \leq \alpha \leq 1 \), in \( (-\pi, \pi) \), if \( f(x) \in L_p \), and

\[
\int_{-\pi}^{\pi} \left| f(x + h) - f(x - h) \right|^p dx = O(h^{\alpha p}).
\]

The functions \( \phi_n(t), (n = 0, 1, 2, \cdots) \), are the Rademacher functions.§

* Hardy and Littlewood, Some properties of fractional integrals I, Mathematische Zeitschrift, vol. 27 (1927–28), pp. 565–606. We will refer to this paper as (HL).


‡ Hardy and Littlewood, A convergence criterion for Fourier series, Mathematische Zeitschrift, vol. 28 (1928), pp. 612–634, in particular, §2 and §3.5. See also (Z), §7.4.

§ For definitions and properties see (Z), §§1.32 and 5.5 to 5.61.
THEOREM 1. Let \( c_0, c_1, c_2, \cdots, c_n, \cdots \) be a sequence of real or complex numbers such that
\[
\sum_{n=2}^{\infty} n^{2\alpha} |c_n|^2 (\log n)^{1+\epsilon}
\]
converges for an \( \epsilon > 0 \). Then, for almost all values of \( t \), the function
\[
f_t(x) = \sum_{n=0}^{\infty} c_n e^{i\pi x} \phi_n(t)
\]
belongs to the class \( \text{Lip}(\alpha) \), \((0 \leq \alpha \leq 1)\). The theorem is false for the case \( \alpha = 0 \) or \( \alpha = 1 \) if \( \epsilon = 0 \).

As a consequence of the theorem of Paley and Zygmund mentioned above it follows that
\[
f_t^\alpha(x) = i^\alpha \sum_{n=1}^{\infty} n^\alpha c_n e^{i\pi x} \phi_n(t),
\]
for almost all values of \( t \), is a continuous function (since the series converges uniformly we put \( f_t^\alpha(x) \) equal to the sum of the series). That is, we have
\[
f_t^\alpha(x) \in \text{Lip}(0).
\]
If by the symbol \( f_{t, \alpha}^\alpha(x) \) we denote the integral of \( f_t^\alpha(x) \) of order \( \alpha \), we have
\[
f_{t, \alpha}^\alpha(x) \in \text{Lip}(\alpha).
\]
But
\[
f_{t, \alpha}^\alpha(x) = i^\alpha \sum_{n=1}^{\infty} \frac{c_n n^\alpha}{(in)^\alpha} e^{i\pi x} \phi_n(t)
\]
\[
= f_t(x) - c_0 \phi_0(t).
\]
To show that the theorem is not true in the case \( \alpha = 1 \), for \( \epsilon = 0 \), we consider the function

* (Z), §§9.80 and 9.81. A function satisfies a condition \( \text{Lip}^* (\alpha) \) or \( \text{Lip}^* (\alpha, \rho) \) when it satisfies a condition analogous to that for \( \text{Lip}(\alpha) \) or \( \text{Lip}(\alpha, \rho) \) but with \( \epsilon \) small in place of \( O \) large. In each of our theorems \( \text{Lip}(\alpha) \) or \( \text{Lip}(\alpha, \rho) \) may be replaced by \( \text{Lip}^* (\alpha) \) or \( \text{Lip}^* (\alpha, \rho) \), respectively, except in the case \( \alpha = 1 \); this follows from Theorems 18, 21, and 22 of (HL).
This can not belong to Lip(1) for any sequence of signs since it is the integral of
\[ \sum_{m=1}^{\infty} \frac{\pm i^{m+1}}{2^m m \log (m + 1)}. \]

which is Paley and Zygmund’s example* of a series which does not represent a bounded function for any sequence of signs.

For the case of Lip (α, p), (0 ≤ α ≤ 1, p ≥ 1), we have a similar theorem.

**Theorem 2.** Let \( c_0, c_1, c_2, \ldots, c_n, \ldots \) be a sequence of real or complex numbers such that \( \sum_{n=1}^{\infty} n^{2\alpha} |c_n|^2 \) converges. Then, for almost all values of \( \alpha \), the function
\[ f_\alpha(x) = \sum_{n=0}^{\infty} c_n e^{in\alpha} \phi_n(t) \]

belongs to the class Lip (α, p), (p ≥ 1, 0 ≤ α ≤ 1).

Since \( \sum_{n=1}^{\infty} |n^\alpha c_n|^2 \) is convergent, it follows that†
\[ f_\alpha(x) = i^\alpha \sum_{n=1}^{\infty} n^\alpha c_n e^{in\alpha} \phi_n(t) \]

belongs to \( L_p, (p \geq 1) \).

Now by a theorem of Hardy and Littlewood‡
\[ f_{\alpha, \epsilon}(x) \in \text{Lip} (\alpha, \epsilon) \]

But, as before,
\[ f_{\alpha, \epsilon}(x) = f_\alpha(x) - c_0 \phi_0(t) \]

As a corollary of the following theorem we have a better theorem for the case 1 ≤ p ≤ 2.

* Paley and Zygmund, loc. cit., p. 350. This gives the case \( \alpha = 0, \epsilon = 0 \).
† (Z), §5.6 (iii).
‡ (HL), Theorems 21, 22, and ff.
THEOREM 3. If
\[
\sum_{n=-\infty}^{+\infty} n^{p'\alpha} |c_n|^p,
\]
converges, then
\[
f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{i\alpha}
\]
belongs to the class Lip \((\alpha, p), \ 1/p + 1/p' = 1, \ (0 \leq \alpha \leq 1)\).

From the Young-Hausdorff* theorem we have
\[
\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(\alpha)}(x)|^{p'} dx \right)^{1/p'} \leq \left( \sum_{n=-\infty}^{+\infty} |n^{\alpha}c_n|^{p'} \right)^{1/p'},
\]
where
\[
f^{(\alpha)}(x) \sim \sum_{n=-\infty}^{+\infty} n^{\alpha}c_n e^{i\alpha x}.
\]
Since \(f^{(\alpha)}(x) \in L_p\), we have†
\[
f(x) - c_0 = f^{(\alpha)}(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\alpha x} \in \text{Lip} \ ((\alpha, p), \ (p \geq 2)).
\]

COROLLARY. If
\[
\sum_{n=-\infty}^{+\infty} n^{2\alpha} |c_n|^2,
\]
converges, then
\[
f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{i\alpha x}
\]
belongs to Lip \((\alpha, p)\) for every \(p\) such that \(1 \leq p \leq 2\).

This follows because
\[
\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^{p} dx \right)^{1/p} \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^{2} dx \right)^{1/2}.
\]

BROWN UNIVERSITY

* (Z), §9.1.
† (HL), Theorem 21.