ZYGMUND ON TRIGONOMETRIC SERIES

Trigonometric Series. By Antoni Zygmund. Warszaw, Monografje Matematyczne, Volume V, 1935. iv+320 pp.

It has been a repeated privilege of the reviewer to express his appreciation of the high standards and excellent quality of the series of monographs of which Volume V is now under his consideration. Each volume of the series published so far represents an important event in the development of mathematical research, and the present volume in this respect is second to none of its predecessors. If one looks through the long list of books on Fourier series one can not help feeling that even the bulkiest of them are far from giving an adequate picture of the present status of the field. The non-existence of a monograph giving such a picture was very badly felt not only by the beginners but also by specialists, and the failure of so many attempts to write a real book on Fourier series created an impression that the task was almost hopeless. The author of the present monograph completely succeeded in dispelling this "inferiority complex" and produced a book which not only introduces the reader into the immense field of the theory of Fourier series but at the same time almost imperceptibly brings him to the very latest achievements, many of them being due to the author himself. The style of the book is rigorous and vigorous and the exposition elegant and clear to the smallest details.

Without wasting his and the reader's time on unessential things the author endeavors to treat each special problem by methods throwing light on the problem from a general standpoint, and showing the place occupied by it in the whole structure. Such a method of exposition will prove to be extremely helpful to a neophyte and will delight a specialist.

Although Zygmund's monograph is far from being the bulkiest of all books written on Fourier series, it certainly contains the richest material. In fact there are but very few topics of importance omitted in it. Thus it is futile to attempt to present here an adequate idea of the subjects treated in the book, and we shall have to restrict ourselves to a very brief description by chapters. Chapters 1 (Trigonometrical series and Fourier series) and 2 (Fourier coefficients, tests for the convergence of Fourier series) are of an introductory character. However, even at this stage the author gives a rather complete discussion of various convergence tests and their mutual relationships, of the order of Fourier-Lebesgue and Fourier-Riemann coefficients, and of operations on Fourier series. Chapter 3 (Summability of Fourier series) contains a rapid but inclusive and elegant discussion of Cesàro and Abel summability of Fourier and Fourier-Stieltjes (derived) series and their conjugates. Chapter 4 (Classes of functions and Fourier series) deals with necessary and sufficient conditions which have to be satisfied by the Fourier series of a function in order that this function should belong to a certain specified function space (such as L_p, and a more general space L_ϕ, continuous, bounded and measurable, and the like). The generalized Parseval identity and the theory of factor sequences transforming one class of functions into another find their appropriate place here. A systematic use of notation and results of the theory of linear operation in abstract spaces and of general inequalities based on the theory of convex functions appear here as natural and powerful tools. Chapter 5 (Properties of some special series) treats of topics like convergence factors, lacunary series, and certain power series as a source of "Gegenbeispiele" of various kinds. Considerable use is made of the theory of Rademacher series. The subject of Chapter 6 (The absolute convergence of Fourier series) is clear from its title. Aside from the classical results of Fatou, Lusin, and Denjoy, it contains a discussion of exponents of convergence of Fourier coefficients of functions satisfying Lipschitz conditions, of the absolute convergence of some lacunary series, and closes with a recent important theorem of Wiener and its extensions. Chapter 7 (Conjugate series and complex methods in the theory of Fourier series) contains a wealth of material concerning the conjugate of a Fourier series. Extremely simple proofs of fundamental inequalities of M. Riesz and of a theorem of Hardy and Littlewood, and Fejér deserve a particular mention. Chapter 8 (Divergence of Fourier series, Gibbs' phenomenon) contains, among many other important topics, a treatment of Gibbs' phenomenon for Cesáro sums of a Fourier series (Cramér's theorem) and an exposition of the famous example of an everywhere divergent Fourier series of Kolmogoroff. Chapter 9 (Further theorems on Fourier coefficients, integration of fractional order) is devoted to an exhaustive exposition of the theory of Fourier coefficients, which originated in the classical theorem of Young-Hausdorff-F. Riesz, and was developed by Hardy and Littlewood, Paley, and the author himself. The comparatively recent but already classical theorem of M. Riesz concerning the convexity of moduli of linear transformations in abstract spaces is of course the main tool here. For lack of space we omit the description of the very interesting Chapter 10 (Further theorems on the summability and convergence of Fourier series) and pass on to the next, Chapter 11 (Riemann's theory of trigonometrical series). This chapter is a masterpiece of concise and clear exposition. It contains, together with the classical results, an exposition of the theory of formal multiplication of trigonometric series developed by Rajchman and Zygmund, of uniqueness and multiplicity sets, and, as a final climax, the latest investigations of Verblunsky concerning the uniqueness problem for Abel summable series. The last Chapter 12 (Fourier's integral) gives a very elegant exposition of the Plancherel-Titchmarsh theory of Fourier transforms and of fundamental results of the theory of representation of a function by means of Fourier integrals. This chapter should be considered as an introduction to the theory of Fourier integrals—a vast field whose "adequate treatment would require a separate book," as the author rightly states in his preface.

Each chapter is supplied with numerous "Miscellaneous theorems and examples." They cover considerable ground by themselves, and unlike the usual type of "exercises" of this kind, are supplied with hints and references sufficient to make them more than a collection of puzzles. The book closes with a substantial bibliography. The typography and general setting of the book are excellent, although the number of misprints (including one or two lapses) is more than negligible. On the whole the author should be congratulated upon producing not only a good book but *the* book on Fourier series, the study of which will benefit the beginners and specialists alike through years to come, by bringing within their immediate reach the best of what has been achieved in the theory of Fourier series.

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ZARISKI ON ALGEBRAIC SURFACES

Algebraic Surfaces. By Oscar Zariski. Ergebnisse der Mathematischen Wissenschaften, Volume 3, Berlin, 1935. v+198 pp.

We are facing today, in the birational geometry of surfaces and varieties, more than in any other chapter of mathematics, the sharp need of a thoroughgoing and critical exposition. In a subject reaching out in so many directions, the task is bound to be arduous. Nevertheless it is surely urgent and for two reasons. In the first place, a systematic examination of the positions acquired is destined to be of considerable value in subsequent campaigns. In the second place, the territory already conquered and safely held is exceedingly beautiful and deserves to be admired by tourists and not merely by members of the vigorous, but small, conquering army. To speak less metaphorically, in this quarter of mathematics "cantorian" criticism has not penetrated as deeply as in others. This has resulted in a widespread attitude of doubt towards the science, which it would be in the interest of all to dispel as rapidly as possible. Nothing will contribute more to this worthy end than Zariski's splendid book. It is indeed the first time that a competent specialist, informed on all phases of the subject, has examined it carefully and critically. The result is a most interesting and valuable monograph for the general mathematician, which is, in addition, an indispensable and standard *vade mecum* for all students of these questions.

As is well known, when one endeavors to pass from one-dimensional birational geometry to the higher dimensions, the difficulties multiply enormously. Many results do not extend at all, or if they do, they are apt to assume a far more complicated aspect or else to demand most difficult proofs.

Consider for example these two questions: (a) the reduction of singularities; (b) the extension of the properties of the genus of an algebraic curve. For some time we have had quite complete and satisfactory proofs of the fact that any irreducible algebraic curve is birationally transformable into a non-singular curve in some space, or to a plane curve with simple and harmless singularities. A similar result is certain to hold for surfaces and varieties. Not to speak of varieties where complete obscurity still reigns, the proof for surfaces has given rise to much confusion. In Zariski's monograph we find the first critical and complete survey of the situation ever made. From this survey it appears, incidentally, that the only "certifiable" proof now in existence is R. J. Walker's (Annals of Mathematics, April, 1935).

A similar service has been performed by Zariski as regards the theory of the *irregularity* q, the analog of the genus p for a surface. The genus p of an algebraic curve is susceptible of four unrelated definitions: projective, birational-geometric, transcendental, topological. On passing to surfaces, these definitions give rise to different genera, which are distinct but not wholly independent.