to come, by bringing within their immediate reach the best of what has been achieved in the theory of Fourier series.

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ZARISKI ON ALGEBRAIC SURFACES

Algebraic Surfaces. By Oscar Zariski. Ergebnisse der Mathematischen Wissenschaften, Volume 3, Berlin, 1935. v+198 pp.

We are facing today, in the birational geometry of surfaces and varieties, more than in any other chapter of mathematics, the sharp need of a thoroughgoing and critical exposition. In a subject reaching out in so many directions, the task is bound to be arduous. Nevertheless it is surely urgent and for two reasons. In the first place, a systematic examination of the positions acquired is destined to be of considerable value in subsequent campaigns. In the second place, the territory already conquered and safely held is exceedingly beautiful and deserves to be admired by tourists and not merely by members of the vigorous, but small, conquering army. To speak less metaphorically, in this quarter of mathematics "cantorian" criticism has not penetrated as deeply as in others. This has resulted in a widespread attitude of doubt towards the science, which it would be in the interest of all to dispel as rapidly as possible. Nothing will contribute more to this worthy end than Zariski's splendid book. It is indeed the first time that a competent specialist, informed on all phases of the subject, has examined it carefully and critically. The result is a most interesting and valuable monograph for the general mathematician, which is, in addition, an indispensable and standard *vade mecum* for all students of these questions.

As is well known, when one endeavors to pass from one-dimensional birational geometry to the higher dimensions, the difficulties multiply enormously. Many results do not extend at all, or if they do, they are apt to assume a far more complicated aspect or else to demand most difficult proofs.

Consider for example these two questions: (a) the reduction of singularities; (b) the extension of the properties of the genus of an algebraic curve. For some time we have had quite complete and satisfactory proofs of the fact that any irreducible algebraic curve is birationally transformable into a non-singular curve in some space, or to a plane curve with simple and harmless singularities. A similar result is certain to hold for surfaces and varieties. Not to speak of varieties where complete obscurity still reigns, the proof for surfaces has given rise to much confusion. In Zariski's monograph we find the first critical and complete survey of the situation ever made. From this survey it appears, incidentally, that the only "certifiable" proof now in existence is R. J. Walker's (Annals of Mathematics, April, 1935).

A similar service has been performed by Zariski as regards the theory of the *irregularity* q, the analog of the genus p for a surface. The genus p of an algebraic curve is susceptible of four unrelated definitions: projective, birational-geometric, transcendental, topological. On passing to surfaces, these definitions give rise to different genera, which are distinct but not wholly independent.

These invariants were extensively investigated by Castelnuovo, Enriques, Picard, Severi, and in certain relatively simple cases, they enable one to separate certain important classes of surfaces (such as rational or hyperelliptic surfaces, and reguli). The central, and also most easily described result is the following: *Let q be the irregularity of an algebraic surface F ($2q$ = first Betti number of F) and let $|C|$ denote a complete linear system of algebraic curves in F. Then $|C|$ is a member of an algebraic family whose dimension never exceeds and generally equals q.* The possible dimensions may be given when the structure of the period-matrix of the Picard integrals (linear integrals) of the first kind is known. This fundamental theorem was obtained by the combined efforts of the above geometers together with Humbert and Poincaré. Zariski's careful analysis has shown that the one solid proof is Poincaré's, which is last in date, and wholly analytical. Indeed the perusal of Zariski's book shows clearly that in this domain the most solid ground is always to be found wherever analysis and topology predominate, for example in the theory of the base (investigations of Picard, Severi, and Lefschetz). It serves also to confirm the reviewer's opinion of many years standing, that it would be most desirable to develop the whole theory from the function-theoretic point of view, as a chapter in the general theory of analytic functions of several variables. For that matter, this should likewise be done from the point of view of modern algebra, although in this direction there is some danger of losing much of the geometric charm.

The preceding remarks risk leaving the reader with the unjust impression that Zariski has occupied himself chiefly with a critique of things long since done. As a matter of fact, this critique takes up only half of his monograph, the rest being given over to modern questions. Thus the topological structure of a surface is treated with abundant care; the recent and very striking results of Hodge are duly described; likewise there is a summary of the theory of punctual series of equivalence on surfaces recently developed largely by Severi, together with its contacts with the reviewer's topological fixed point theory. Altogether we cannot recommend this monograph too strongly to all lovers of geometry in every form. They will leave it convinced of the beauty and vitality of this most attractive branch of modern mathematical research.

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