ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Dr. Ralph Hull: *The maximal orders of generalized quaternion division algebras.*

A generalized quaternion division algebra Q is of the form $Q=(a, Z)=Z+uZ$, where Z is a quadratic field, $u^2=a$ is rational and not the norm of an element of Z, and $Zu=uZ'$ elementwise. In this paper for each Q all maximal orders $\mathfrak{M}$, that is, maximal sets of integral elements, are determined explicitly. For any maximal order $\mathfrak{M}$ of Q, let $\mathfrak{m}_c$ be the intersection of $\mathfrak{M}$ and Z. Then $\mathfrak{m}_c$ is an order in Z with a positive rational integral conductor c, and there exists an $\mathfrak{m}_c$-ideal $\mathfrak{n}$ of Z, not necessarily either integral or regular, and a quantity λ of Z, such that $\mathfrak{M}=\mathfrak{m}_c+(\lambda+u)\mathfrak{n}$. Conversely, in terms of a special generation for each Q found by Albert (this Bulletin, vol. 40 (1934), pp. 164–177) all $\mathfrak{m}_c$, and for a fixed c, all $\mathfrak{n}$ and λ, such that $\mathfrak{m}_c+(\lambda+u)\mathfrak{n}$ is an $\mathfrak{M}$, are determined For a fixed c there are infinitely many $\mathfrak{n}$, and for each of these a rational basis is easily found. For fixed c and $\mathfrak{n}$, the finite number of λ which yield distinct $\mathfrak{M}$ are determined. A single rational basis for each $\mathfrak{M}$ is determined. (Received November 1, 1935.)

2. Professor H. J. Ettlinger and Mr. O. H. Hamilton: *Sets of functions and their limit functions.*

For sets of functions (real) on I: $0\leqq x\leqq 1$, the following theorems are proved: 1. Extension of Arzela's theorem on the necessary and sufficient condition that a set of functions converge to a continuous limit function to include absolutely continuous functions which converge to absolutely continuous limit functions. 2. Extension of Ascoli's theorem to the effect that a bounded equicontinuous set of functions has a subsequence which converges to a continuous limit function, and the corollary of Graves that if the set above is equi-absolutely continuous, the limit function will be absolutely continuous, to include the result that the original set plus all of its limit functions will be equi-absolutely continuous. 3. Any uncountable (or absolutely continuous) set of functions on I contains a bounded subsequence which has a continuous (or absolutely continuous) limit function. 4. If $f^\alpha(x)$ is a collection of continuous functions on I and no two of these functions cross, there is a subsequence $f_n(x)$ such that each point which lies on two or more functions $f^\alpha(x)$ lies on $f_n(x)$ for

some natural number n. 5. For every bounded equi-continuous collection such that the sum of a finite number of pieces itself belongs to the collection, the upper (and lower) bound functions of the set are continuous functions and are approached uniformly as limits by subsequences of the given set. (Received November 4, 1935.)

3. Mr. P. M. Pepper: *Application of geometry of numbers to a generalization of continued fractions.*

In Minkowski's Collected Works, vol. 1, p. 279, appears a paper entitled *Zur Theorie der Kettenbruche*. There a generalization of a continued fraction, taking the form of a set of square matrices of integers, is exhibited along with an algorithm for computing these matrices. Several theorems upon which this generalization depends are stated without proof. Let ξ, η, ζ be linear homogeneous forms in the three independent real variables x, y, z having real coefficients and such that no one of these forms vanishes for any triple of integers not all zero. There exists an infinity of triples of integers x, y, z for which $|\xi\eta\zeta| < |\Delta|$ where Δ is the determinant of the three forms. Such triples can be secured for which any desired one of these linear forms is arbitrarily small in absolute value. To ξ, η, ζ are made to correspond square matrices, each consisting of three such triples, these matrices making a chain such that to each matrix there correspond three "neighboring" matrices. This chain is analogous to the sequence of convergents of a continued fraction. In this paper proofs of the above theorems are given with a derivation of the algorithm mentioned. An example will also be given. (Received November 4, 1935.)

4. Professor H. B. Curry: *A note on the associative law in logical algebras.*

A proof of the redundance of the associative law in the Principia Mathematica was published by Bernays in 1925. The present note gives an alternative proof, which is applicable to a whole class of quasi-logical systems. This proof is a refinement of one due to C. S. Peirce. (Received November 4, 1935.)

5. Professor Hassler Whitney: *Differentiable functions defined in arbitrary subsets of euclidean space.*

Let A and B be arbitrary subsets of n-space E_n. Let $f_k(x)$, $(\sigma_k = k_1 + \cdots + k_n \leqq m)$, be defined in A. If the remainders $R_k(x'; x)$ in the Taylor formulas are defined as in the Transactions of this Society, vol. 36 (1934), pp. 63–89, (3.1); and (3.2) holds at x^0 in B for each $\epsilon > 0$; we say $f(x)$ is of class C^m in A about x^0 in terms of the $f_k(x)$: $f(x)$ is $(C^m, A, x^0, f_k(x))$. If this is so at each x^0 in B, $f(x)$ is (C^m, A, B). It is shown that if $f(x)$ is (C^m, A, B), then it may be extended throughout E_n so that it is (C^m, E_n, B), generalizing Lemma 2 of the above paper; then in any open subset of B, $f(x)$ has continuous mth derivatives. If $f(x)$ is (C^m, A, B), then there is an open set $B' \supset B$ such that $f(x)$ is (C^{m-1}, A, B'). Functions of functions of class C^m are of class $C^m \cdots$. Finally, the above properties are local: If each p in B is in a neighborhood U such that $f(x)$ is $(C^m, A \cdot U, B \cdot U)$, then $f(x)$ is (C^m, A, B). (Received November 7, 1935.)

6. Professor Francis Regan: *On the admissibility of time series.*

In a paper by the author (Transactions of this Society, vol. 36 (1934), pp. 511–529) a time series is constructed in such a manner that the number $x(\alpha, \tau, t, \Lambda)$ is an element of the set $A[f(\alpha, \tau, t)]$, for every α, τ, t, and Λ, where $\tau = m \cdot 2^{-\sigma+1}$, $t = r \cdot 2^{-\sigma+1}$, $\Lambda = \rho \cdot 2^{-\sigma+1}$, and $r+m \leqq \rho$; and where α, ρ, m, r, and σ are positive integers. Here it is shown that for the same series the number $x(\alpha, \tau, t, \Lambda)$ is a member of the set $A[f(\alpha, \tau, t)]$, for every α, τ, t, and Λ, where $\Lambda = \rho \cdot 2^{-\sigma+1}$ and $t+\tau \leqq \Lambda$, with α, ρ, and σ as positive integers. (Received November 13, 1935.)

7. Mr. W. H. Ingram: *On the dynamics of commutator machines.*

The discovery of the analogy between relative angular velocity of commutator and brush in the theory of a commutated electrical nexus and the angular velocity of a system of moving coordinates in dynamics (Abstract 40-1-23, Philosophical Magazine, April, 1934, p. 844) suggests the possibility of establishing, for the electrical circuits in a machine, an analogy to the inertial ellipsoid of Cauchy and Poinsot in particle dynamics. It is found that the ellipsoid is stationary with respect to the field and that two principal diameters coincide with the polar and inter-polar axes, respectively. Suitable quasi-coordinates for commutator machines of any complexity are obtained from the true coordinates by a rotation of the coordinate system about an $(n-2)$-space perpendicular to the plane of the polar and inter-polar axes. (Received November 8, 1935.)

8. Mr. E. H. Larguier: *On a method for evaluating the moments of a Bernoulli distribution.*

This paper gives the moments of a Bernoulli distribution, $(p+q)^n$, in the form of a polynomial in p (the probability that the event will occur in a single trial) in which the coefficients are functions of n and are determined by a recursion formula. (Received November 15, 1935.)

9. Dr. E. P. Northrop: *Note on a singular integral.* II.

This paper is a generalization of the results obtained by the author in an earlier note (this Bulletin, vol. 40 (1934), pp. 494–496), in which were obtained necessary and sufficient conditions for the convergence in the mean to $f(x)$, as $m \to \infty$, of the integral $(2\pi)^{-1/2}\int_{-\infty}^{+\infty} f(u)K(x-u; m)du$, where $f(x)$ is an arbitrary function of L_2, and $K(x; m) \epsilon L_2$ for every m. The main results of the present note are concerned with sufficient conditions in the case $f(x) \epsilon L_q$, $2 < q < \infty$; and with necessary conditions in the case $f(x) \epsilon L_p$, $1 < p < 2$, and $K(x; m)$ is for every m the Fourier transform in L_q $(1/p+1/q=1)$ of some function in L_p. The simplifications brought about in the conditions on the kernel when it is of the special form $K(x; m) = mk(mx)$ are also discussed. (Received November 15, 1935.)

10. Professor A. H. Copeland (Guggenheim Fellow): *A critique of the continuum.*

It is the contention of this paper that the conventional postulational sys-

tems for the continuum are inconsistent. The inconsistency can be avoided by the introduction of a canon for postulation which prescribes certain alterations of these systems. When such changes have been effected, it becomes obvious why the original schemes are self-contradictory. Most of the classical results can be obtained from the altered set of postulates. There are, however, the following notable exceptions: the Richard paradox is resolved, a proof is given that the continuum cannot be well-ordered, and Zermelo's axiom is shown to be untenable. (Received November 15, 1935.)

11. Mr. W. E. Sewell: *Location of the level curves of Green's function.*

Let $w=f(z)$ map the exterior of the unit circle in the z plane conformally on the exterior of a simply connected region R in the w plane, carrying the point $z=\infty$ into the point $w=\infty$. This mapping carries the circle $|z|=\rho>1$ into an analytic Jordan curve C_ρ, and as ρ approaches 1 the curve C_ρ approaches C, the boundary of R. The minimum distances of a point on C from the closed set C_ρ and of a point on C_ρ from the closed set C depend upon ρ and C. In this paper it is shown that each of these distances is bounded by $M(\rho-1)^\mu$, where M and μ are constants depending only on C. A typical result is the following: Let $w=f(z)$ map the exterior of the unit circle on the exterior of a Jordan curve C in the w plane, carrying the point $z=\infty$ into the point $w=\infty$. Then for $1<\rho<2$ no point of C lies closer to C_ρ than $M(\rho-1)^2$, M depending only on C. (Received November 15, 1935.)

12. Mr. Marshall Hall: *A note on null sequences.*

A linear recurring sequence is a null sequence modulo m if every term after a certain point is divisible by m. A sequence is said to be p-adically null if it is a null sequence modulo every power of p. It is shown that a necessary and sufficient condition for a sequence to be p-adically null is that p divide every coefficient of the recurrence. It is, however, possible to construct a sequence satisfying a given recurrence, whose last coefficient is divisible by p, which shall be null modulo p^i in a non-trivial manner. (Received November 18, 1935.)

13. Dr. Nilan Norris: *Convexity properties of generalized mean value functions.*

Some investigations concerning the behavior of the second derivatives of four of the more important types of generalized mean value functions indicate that, although deductions as to convexity properties of the various means as functions of t must be very carefully guarded, certain inferences may be drawn. These conclusions are useful in applications of properties of the chief types of mean value functions to statistical theory, especially to index number theory. (Received November 18, 1935.)

14. Professor I. A. Barnett and Dr. C. W. Mendel: *On a certain integral equation quadratic in the unknown function.*

The problem considered in this paper is to find all real valued continuous functions $y(x)$ defined on $0\leqq x\leqq 1$, and satisfying the integral equation

$y^2=(y-\kappa)\int_0^1 y\,dx$, where $\kappa(x)$ is a prescribed continuous function on this interval. It is shown that the solution of the given equation may be made to depend upon a step function $\epsilon(x)$ taking on only the values $+1$ and -1 on $(0, 1)$. A step function is said to be of type r if it has precisely $r-1$ discontinuities none of which is isolated. A solution is said to be of type r if the associated step function is of type r. The principal result of this paper is as follows: Let M and m denote, respectively, the absolute maximum and minimum of $\kappa(x)$ on $(0, 1)$. Let $\lambda(c)=\int_0^1[(1-\sqrt{1-c\kappa})/c]dx$. Then, necessary and sufficient conditions that there shall exist a unique, real valued, continuous solution $y(x)$ of type 1, are that (1) $\kappa(x)$ changes sign on $(0, 1)$, (2) $\int_0^1 \kappa(x)dx\neq 0$, and (3) $\lambda(1/M)\geqq 0\geqq\lambda(1/m)$. As an illustration, it turns out that for the function $\kappa(x)\equiv a(x-\alpha)$, $(0<\alpha<1)$, the integral equation will have a solution of the type described if, and only if, α lies on the middle ninth of the interval $(0, 1)$ excluding $\alpha=1/2$. (Received November 19, 1935.)

15. Mr. Garrett Birkhoff: *A note on topological groups.*

It is proved that a topological group (or "Hausdorff" group) satisfying Hausdorff's first countability axiom, is metrisable. This was previously known only under assumption of Hausdorff's second countability axiom. (Received November 20, 1935.)

16. Professor R. V. Churchill: *Temperature distribution in a slab of two layers.*

The Laplace transformation is used to find the variable temperature distribution formula for a slab of finite thickness consisting of two layers of different materials, when the initial temperature distribution is arbitrary. The purpose of the paper is not only to derive a new temperature formula, but also to illustrate a procedure by which the Laplace transformation and its inverse can be used in such problems in a dependable manner. The Mittag-Leffler partial fractions expansion and contour integrals are used in establishing the inverse transformation. (Received November 20, 1935.)

17. Professor J. D. Mancill: *The minimum of a definite integral with respect to unilateral variations.*

This paper is devoted to a rather exhaustive treatment, by comparatively general methods, of both necessary conditions and sufficient conditions for a strong relative minimum with respect to unilateral variations in plane problems in the calculus of variations, *without the hypothesis of regularity*. The analogous problem in space of three dimensions is also treated but the results are not as exhaustive. Both necessary conditions and sufficient conditions are obtained for the space problem but only under rather restrictive hypotheses. This fact and the fact that many of the difficulties encountered in the two problems differ widely, make it desirable to treat them separately. (Received November 20, 1935.)

18. Professor Dunham Jackson: *Orthogonal polynomials in two variables: formal properties.*

This paper is concerned with the construction and formal properties of

systems of orthogonal polynomials in two variables, which present some non-trivial points of novelty in comparison with the corresponding theory in one variable, particularly in connection with the orthogonal transformation of the set of polynomials of specified degree induced by a linear transformation of the variables under which the region of definition and the weight function are invariant. (Received November 21, 1935.)

19. Dr. W. C. Randels: *On the summability of Fourier series.*

Necessary and sufficient conditions for L, L', $\tilde{L}$, and $\tilde{L}'$-effectiveness of regular methods of summation of a general type are given. This type includes all triangular methods. (Received November 21, 1935.)

20. Mr. Alvin Sugar: *On universal Waring theorems for cubic functions.* Preliminary report.

This is the second of a series of papers in which the author hopes to verify some special cases of the following conjecture. Consider the polynomial $F_n(x) = M_{n-1}[x, n] + \cdots + M_1[x, 2] + x$, where $M_{n-1}, \cdots, M_1$ are non-negative integers and $[x, k] = (x+k-2)\cdots x(x-1)/k!$ (we may, by analogy, speak of the value of $F_n(x)$ for $x \geqq 0$ as nth degree polygonal numbers). In this paper the writer makes the conjecture that every positive integer is a sum of $s = \sum_{i=1}^{n-1} M_i + n$ values of $F_n(x)$ for $x \geqq 0$. It is to be noted that for $M_{n-1} \geqq \sum_{i=1}^{n-3} [n-(i+2)]M_i + 2n - 3$ such theorems would be ideal universal Waring theorems, i.e., there exists an integer $I(M_i, n)$ which is not a sum of fewer than s values. This paper establishes such theorems for the polynomials $(x^3-x)m/6 + (x^2-x)n/2 + x$, where m divides n. The methods are similar to those of a previous paper by the author entitled *A cubic analogue of the Cauchy-Fermat theorem.* (See abstract 41-11-410.) The constant employed here is that obtained by Webber. (Received November 21, 1935.)

21. Mr. H. S. Zuckerman: *New results for the number $g(n)$ in Waring's problem.*

This paper is a continuation of the work described in a preliminary report (abstract 41-3-163). Since then Vinogradow has published (Annals of Mathematics, vol. 36 (1935), pp. 395–405) another proof that $G(n) \leqq s = [n(6 \log n + \log 216 + 4)]$. The constants in this new proof are evaluated to obtain a value N such that all integers greater than N can be expressed as the sum of s positive or zero integral nth powers. A method for treating the integers greater than zero and less than $N+1$ without the use of tables is developed. As an illustration, ideal universal Waring theorems are proved for $h = 15, 16, 17, 18, 19, 20$; that is, the formula $g(n) = 2^n + [(3/2)^n - 2]$ is verified for these values of n. (Received November 21, 1935.)

22. Dr. R. D. James: *The number of representations of an integer as a sum of twelve, sixteen, or twenty squares.*

Let $N(2^\alpha m, 4s+4)$ denote the number of representations of an integer $2^\alpha m$, $\alpha \geqq 0$, m an odd integer >0, as a sum of $4s+4$ squares. Let $\sigma_{2s+1}(m) = \sum d^{2s+1}$, where the sum is over all divisors d of m. The problem of the pres-

ent paper was suggested by the fact that $N(2^\alpha m,\ 12)=c_5(\alpha)\sigma_5(m)$, if $\alpha\geqq 1$, *but not if* $\alpha=0$. It was thought that a similar result might hold for $N(2^\alpha m,\ 4s+4)$ for $s\geqq 3$. The main result of the paper is that this is not so when $s=3$ or 4. It is proved that for $s=3$ or 4 and any $\alpha\geqq 0$ no relations of the form $N(2^\alpha m,\ 4s+4)=c_{2s+1}(\alpha)\sigma_{2s+1}(m)$ can hold for all integers m with $c_{2s+1}(\alpha)$ independent of m. (Received November 21, 1935.)

23. Professor Nathan Altshiller-Court: *On the Cevian tetrahedron.*

If the lines joining the point M to the vertices of the tetrahedron $(T)\equiv ABCD$ meet the respectively opposite faces of (T) in the points A', B', C', D', the tetrahedron $(T')\equiv A'B'C'D'$, the "Cevian tetrahedron of M for (T)," is obviously homological to (T). It is shown that (i) the plane of this homology is the tetrahedral polar plane of the point M with respect to both (T) and (T'); (ii) the anharmonic ratio of this homology is equal to -3, irrespective of the choice of M. If we construct, in turn, the Cevian tetrahedron (T'') of M for $(T'),\cdots$, the anharmonic ratio of any four consecutive tetrahedrons is constant and equal to 4/7. (Received November 21, 1935.)

24. Professor V. W. Adkisson: *On the groups associated with certain cyclic curves.* Preliminary report.

This note deals with groups of transformations associated with each of the cyclicly connected continuous curves on a sphere having only a finite number of simple closed curves such that the complementary domain boundaries are homeomorphic. These transformations are analogous to the groups associated with the regular polyhedra. Each curve is unique, and has a unique group except one. This curve, in which each region boundary has three branch points of order four and one of order three, may have one of two groups of transformations; the octahedral group, or the dihedral group of order eight. If reversal of sense on the region boundaries be permitted, the octahedral group is extended to a group of order 48, generated by P and Q, $P^2=Q^3$, $(PQ)^4=1$. The dihedral group is extended to a group of order 16 generated by S and T, $S^8=T^2=1$, $T^{-1}ST=S^{-1}$. (Received November 23, 1935.)

25. Professor H. J. Ettlinger: *Cauchy fields of ordinary differential equations.*

Let R be the set of points, (x, y), where x is on $(0, 1)$ and all real values of $y\equiv(y_1,\cdots,y_m)$, and let $f(x, y)$ be a real function defined in R. If $f(x, k)$, where k is a constant, is integrable (Lebesgue) and (t_{in}) defines a fine net on $(x_0, 1)$, we will call the uncountable collection of functions $y_n(x)=y_{i-1,n}+\int_{t_{i-1,n}}^{x} f(t,y_{i-1,n})dt$ on $(t_{i-1,n}, t_{in})$, $y_{0n}\equiv y_0$, $x_{0n}\equiv x_0$, the Cauchy field of $y'=f(x, y)$ through the point (x_0, y_0) of R. It is proved that (i) there exists a subsequence of the Cauchy field which converges to an absolutely continuous limit function, (ii) if this subsequence is equi-absolutely continuous and $f(x, y)$ is continuous in y, the limit function satisfies the equation (1): $y(x)=y_0+\int_{x_0}^{x} f(t,y)dt$, (iii) the Osgood property is established, namely, the upper and lower bound functions of the

field are solutions of (1) and through every point between them there is at least one solution of (1). Related properties are established for the non-unique solutions of (1). (Received November 23, 1935.)

26. Mr. O. H. Hamilton: *Non-unique solutions of first order ordinary differential equations.*

By making use of results announced by Professor H. J. Ettlinger, theorems concerning the properties with respect to continuity of the non-unique solutions of first order ordinary differential equations are demonstrated under hypotheses assumed by Professor Ettlinger. It is shown that under these hypotheses a differential equation, $y'=f(x, y)$, having a non-unique solution at a point P can be arbitrarily approximated by a differential equation having a unique solution at that point. It is further shown that $y'=f(x, y)$ may be arbitrarily approximated by a differential equation having a unique solution over a countable dense set of points in the region. The diameter of a non-unique solution is defined, and it is shown that the set of points in the region which lie on non-unique solutions of diameter greater than a given constant, K, is a closed set of points and that the set of all points which lie on non-unique solutions is the sum of a countable number of closed point sets. (Received November 23, 1935.)

27. Professor C. O. Oakley: *On semi-linear equations.*

In this paper the author continues his study of semi-linear equations, i.e., equations of the form $u_0+\sum m_i|u_i|=0$, where the u's are linear forms in two variables and the m's are constants. Special stress is put upon equations whose solutions are real in character. The general treatment considers the determination of the semi-linear equation which represents each of p regions of the plane formed by n lines. (Received November 23, 1935.)

28. Professor R. G. Archibald: *Highly composite ideals.*

In an earlier paper the author obtained results on the highly composite numbers defined in 1915 by S. Ramanujan. The present paper defines an ideal different from the unit ideal as highly composite if the number of its divisors is greater than that of every ideal of smaller norm and not less than that of every ideal of the same norm. Analogues of the properties of highly composite numbers are sought for these highly composite ideals. Among the results obtained are the following: For an arbitrary algebraic number field, it is shown that the reciprocals of the norms of the highly composite ideals form a convergent series. Where n is the degree of the field and x is an arbitrary positive integer, there always exists a highly composite ideal of norm greater than x and not greater than $2^n x$. (Received November 23, 1935.)

29. Dr. J. H. Curtiss: *A note on the degree of polynomial approximation.*

By a demonstration involving theorems of Fejér and Jackson, it is established that if a function $f(z)$ is analytic interior to an analytic Jordan curve C, is continuous in the corresponding closed region, and has an rth derivative satisfying a Lipschitz condition of order $\alpha \leqq 1$ on C, then there exist polynomials

$p_n(z)$, depending on r and α, such that $f(z)-p_n(z)=O(n^{\alpha+r})$, for $r=0, 1, \cdots$. An application is given to the study of Lagrange interpolation in regularly distributed points, and to the problem of determining the degree of approximation of certain Riemann sums closely identified with the Lagrange polynomials. (Received November 25, 1935.)

30. Dr. J. H. Curtiss: *The Jacobi series on an unrestricted lemniscate.*

This paper is a continuation of the study undertaken by the author of the properties of the Jacobi series on the boundaries of its regions of convergence (see abstract 41-1-28). By the use of certain auxiliary functions constructed on the functions $\phi_k(z)$ introduced in the earlier paper and by means of the Jacobi series corresponding to the auxiliary functions, the author again obtains the analogues for the Jacobi series of some familiar theorems concerning the Taylor series on the circle of convergence; but now with special reference to the case in which the lemniscate of convergence has multiple points. A number of interesting irregularities appear in the theory in this case; for instance, abelian theorems break down at multiple points and class H_2 no longer contains class H_∞. The existence of these irregularities is established by examples. (Received November 25, 1935.)

31. Dr. T. L. Downs, Jr.: *Asymptotic lines through a planar point of a surface and lines of curvature through an umbilic.*

At a planar point (one at which all three coefficients of the second fundamental form of the surface vanish) the classical theorems concerning the asymptotic directions and the asymptotic lines fail to hold. Nevertheless, there exists a finite set of possible tangent directions for the asymptotic lines through the point. The present paper shows that there is tangent to a given direction of this set a unique asymptotic line in the most general case, and two distinct asymptotic lines in the next most general case. While there exists a similar set of possible tangent directions for the lines of curvature through an umbilic, it turns out that a given direction of this set has in general either a single line of curvature or an infinite number of lines of curvature tangent to it, depending upon certain definite conditions. (Received November 25, 1935.)

32. Professor Cornelius Lanczos: *A new approximation method in solving linear differential equations with rational coefficients.*

In the method outlined previously (see abstract 41-3-108) further improvements are obtained. Legendre's polynomials are replaced by Tshebisheff's polynomials which permit a more even distribution of the error. The simple algorithms for these polynomial make it possible to estimate the asymptotic behaviour of the coefficients and prove the convergence. Applications to the approximation of the error integral, the integral logarithm, the Gamma function, Bessel's functions, show the rapid convergence of the method. (Received November 25, 1935.)

33. Professor Deane Montgomery and Dr. Leo Zippin: *Many parameter groups in n-space.*

If x is a point in n-space and t a k-dimensional vector, let $T(x; t)$ denote a function with values in the n-space, continuous in x and t simultaneously. Further, suppose that $T[T(x; t_1); t_2] = T(x; t_1+t_2)$. Among other results, we prove that if $k \geqq n-2$ then in order that $T(x; t)$ be topologically equivalent to the k-parameter translation group in n-space, it is necessary and sufficient that for each point x_0 there exist a sphere S containing it and a positive N such that for *all* $|t| > N$ and *all* points x in S, $T(x; t)$ *is not in* S. The theorem is comprehensive for 3-space. For greater values of n and general k, our problem can be shown to be equivalent to an interesting unsolved problem in topological products. (Received November 25, 1935.)

34. Professor C. N. Moore: *On restricted convergence and restricted summability of double series.*

If the partial sums, s_{mn}, of a double series, $\sum u_{mn}$, converge to a limit when m and n become infinite while obeying the restriction $0 < k < m/n < K$, where k and K are constants, the series is said to be restrictedly convergent. Restricted summability for any method of summation which replaces the s_{mn} by a related sequence, σ_{mn}, is defined in an analogous manner. The notion of restricted convergence and restricted summability was first introduced by the author in connection with the study of the summability of the double Fourier series, where it was shown to have special significance (Mathematische Annalen, vol. 74 (1913), pp. 555–572). Later Bochner showed that the notion had similar significance in connection with the Poisson summation formula (Mathematische Annalen, vol. 106 (1932), pp. 56–63). More recently Lösch has shown that a restrictedly convergent double series is not necessarily restrictedly summable by any regular transformation whatsoever, but that it is restrictedly summable by transformations of the Cesàro or Euler-Knopp type (Mathematische Annalen, vol. 110 (1934), pp. 33–53). It is the purpose of the present paper to show that a similar result holds for certain means of Nörlund type. (Received November 25, 1935.)

35. Miss Alta Odoms: *On the logarithmic mean of double Fourier series.*

The purpose of this paper is to establish sufficient conditions for the summability of the Fourier development of an integrable (L) function by Riesz's logarithmic means at a given point (x_0, y_0) to a sum S. Let a_{pq} be the Fourier coefficients of $f(x, y)$, $s_{ij} = \sum_{p=1}^{i} \sum_{q=1}^{j} a_{pq}$, and $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij}/ij$. It is shown that when $f(x, y)$ satisfies certain conditions, then $\lim_{m,n\to\infty}(S_{mn}/\log m \log n)$ exists and is equal to S. The results obtained are generalizations of results contained in a paper by G. H. Hardy (Quarterly Journal of Mathematics, Oxford Series, vol. 2 (1931), pp. 107–112) on the logarithmic mean of a single Fourier series. (Received November 25, 1935.)

36. Professor W. C. Risselman: *On a problem of approximation associated with a periodic Stieltjes fraction.*

The least square problem associated with the continued fraction

$-p_n/(q_n-z)]_1^\infty$, where p_n and q_n are real numbers satisfying $p_n>0$, $p_{n+k}=p_n$, and $q_{n+k}=q_n$ $(n=1, 2, \cdots)$, is studied. For non-real z, the fraction converges and equals $\int_{-\infty}^{\infty}(x-z)^{-1}d\psi(x)$. The monotonic function $\psi(x)$ is examined and a convergence theorem is established. Let $Q_n(x)$ be the denominator of the nth convergent. The convergence proof is made to depend on the fact that $Q_n(x)/(p_1p_2\cdots p_{n+1})^{1/2}$ remains bounded as $n\to\infty$ on the interval or intervals on which the approximation is made. (Received November 25, 1935.)

37. Professor I. J. Schoenberg: *On the zeros of successive derivatives of integral functions.*

J. M. Whittaker (Proceedings of the London Mathematical Society, (2), vol. 36 (1934), pp. 451–469) proved recently the following theorem: Let $f(x)$ be an integral function for which (1) $\overline{\lim}_{r\to\infty}\log M(r)/r<1/2$, and (2) $f^{(n)}(-1)f^{(n)}(1)=0$, $(n=0, 1, 2, \cdots)$. Then $f(x)=0$ identically. Whittaker remarks, by consideration of the function $\sin(x+1)\pi/4$, for which $\overline{\lim}\log M(r)/r=\pi/4>1/2$, that the number 1/2, on the right side of (1), cannot be replaced by any number $>\pi/4$. He conjectures that $\pi/4$ is the best constant in the condition (1). A verification of this conjecture is contained in the following theorem: Let $f(x)$ be an integral function for which (3) $\overline{\lim}_{r\to\infty}\log M(r)/r<\pi/4$, and such that every one of its derivatives $f(x), f'(x), f''(x), \cdots$ vanishes somewhere between -1 and $+1$; then $f(x)=0$ identically. (Received November 25, 1935.)

38. Dr. Leo Zippin: *Dual abelian groups.*

This paper forms a natural complement and generalization of a paper by Professor J. W. Alexander and the author on *Discrete abelian groups and their character groups* in the Annals of Mathematics, vol. 36 (1935). The guiding new theorem is: If A and B are "sufficiently numerous" groups of characters of each other, they determine topologies on each other in which each becomes the group of all continuous characters of the other. (Received November 25, 1935.)

39. Professor J. H. Roberts: *Collections filling a plane.*

The present paper contains the following two results: (1) There exists an upper semi-continuous collection G filling a plane S such that every element g of G is a bounded continuous curve. In fact the element g is either an arc or is the sum of three arcs joined so as to form the letter H. (2) There does not exist an upper semi-continuous collection G of arcs filling a plane S. This second result contradicts the statement which the author presented in April, 1935. (See abstract 41-5-196.) (Received November 25, 1935.)

40. Dr. E. W. Titt: *Cauchy's problem for the non-linear hyperbolic differential equation in three independent variables.*

This paper presents a theory which seems to offer a generalization of the method and the results of H. Lewy in the case of two independent variables (Mathematische Annalen, vol. 98, (1928), p. 179). We use a two-dimensional

characteristic theory (see abstract 41-3-103). In order to carry through the existence theorems we find it sufficient that our initial strip be analytic in one direction but merely having a certain order of regularity in the other direction. These directions are characterized geometrically in terms of the geometry of the characteristic cone. In the course of our work we are led to the consideration of existence theorems for partially analytic solutions of a system of first order partial differential equations. (Received November 25, 1935.)

41. Professor W. M. Whyburn: *Critical sets for functions.*

Difference quotients are used to define critical points for general single-valued functions of n real variables. Critical sets of several types are defined and studied. In the case where the functions are continuous, it is shown that the existence of two critical sets of minimal type implies the existence of a set of minimax type. (Received November 25, 1935.)

42. Professor R. L. Wilder: *Sets which satisfy certain avoidability conditions.*

Relations between certain definitions of i-avoidability at a point (i a dimension number) and of non-i-cut-points are established for various kinds of spaces. As examples of these definitions, a metric space M is *completely i-avoidable* at P if for $\epsilon>0$ there exist δ and η such that $\epsilon>\delta>\eta>0$ and every γ^i of $F(P, \delta)$ is $\backsim 0$ on $S(P, \epsilon)-S(P, \eta)$; *locally i-avoidable* at P if every γ^i of $F(P, \delta)$ is $\backsim 0$ on $M-S(P, \eta)$; P is a *non-i-cut-point* if every γ^i of $M-P$ is $\backsim 0$ on $M-P$. Among the theorems obtained in applying the notions are these (hereafter $i=0, 1, \cdots, n-2$, and sets are in E_n): In order that a simply i-connected compact J^{n-1} (=connected, locally i-connected) should have only simply i-connected *g.c.* $(n-1)-m$.'s as boundaries of its complementary domains, it is necessary and sufficient that it have only non-i-cut-points. Among the sets J^{n-1}, those that have only *g.c.* $(n-1)-m$.'s as boundaries of their complementary domains are characterized by the fact that they are completely j-avoidable for $0\leqq j\leqq n-3$ and locally $(n-2)$-avoidable. The diameters of the complementary domains of a compact J^{n-1} form a null sequence, and the domains are almost all simply i-connected. (Received November 25, 1935.)

43. Dr. D. G. Bourgin: *The problem of the sheet.*

An approximation method has been developed for the problem of the clamped sheet with bending forces neglected. A novelty in the analysis is that the stresses are used rather than the strains. In a sense, an inverse problem has been solved since what is actually accomplished is the determination of the pressure (and attendant stresses) required to deform a square sheet into a given contour. The result involves the solution of an infinite set of linear equations in an infinite number of unknowns. The variation integral for demonstrating uniqueness turns out to be formally identical with the usual one in the strains except as regards sign of one term. The subsequent determination of the strains is given by an equation of hyperbolic type, and it is shown that the boundary conditions are consistent. (Received November 26, 1935.)

44. Dr. D. G. Bourgin: *Group velocity.*

The Rayleigh-Kelvin definition of group velocity may be generalized to phase arguments non-linear in x and t. The generalization is shown to be significant by exhibiting a class of dissipative media defined by a differential equation for which it is proved that the velocity of propagation of the avaiable energy is precisely the generalized group velocity. The demonstration depends on an easy modification of the equation of continuity. In this connection it is shown that the real meaning of the "stationary phase" principle is closely connected to that of the "steepest descent" method, the difference lying solely in the choice of contours. The first utilizes those for which $R.\ P.$ is constant, thus leading to Fourier integral rather than Laplace integral developments. (Received November 26, 1935.)

45. Dr. J. F. Randolph: *Carathéodory linear measure and Vitali's theorem.*

If A is an unrestricted plane set and F is a family of closed sets associated with A under certain conditions involving Lebesgue plane measure, the Vitali theorem states that there exists a sequence of mutually exclusive closed sets selected from F whose union contains all of A except a subset of Lebesgue plane measure zero. With A unrestricted, it has not been proved that under analogous conditions on F involving Carathéodory linear measure, there exists a sequence of mutually exclusive closed sets in F whose union contains all of A except a set of Carathéodory linear measure zero. In this paper it is proved, however, that this conclusion follows for all sets A restricted by two conditions. That these conditions are natural ones is shown by the fact that their analogues for Lebesgue measure are satisfied by all sets A. (Received November 26, 1935.)

46. Professor Henry Blumberg: *On the symmetrical structure of general point sets.*

This paper relates to questions that immediately suggest themselves concerning the symmetrical structure of a general point set in n-space, but which have hitherto not been systematically considered; for example, if S is an unconditioned planar set, what is the nature of the set of points P of the plane such that there are two directions of approach to P, the one having eventually on it only points of S and the other, eventually only points of the complement of S? Such a set of points P is always "ridé," a W. H. Young generalization of "denumerable" to the plane. The paper deals with a variety of analogous questions, and makes application of the results to general functions. (Received November 26, 1935.)

47. Dr. L. M. Blumenthal: *The metric characterization of a certain class of spaces.*

A class of spaces is defined by certain simple postulates and their metric characterization is obtained. The class of spaces includes the euclidean and non-euclidean spaces, and thus is obtained a unified treatment of these spaces, as well as other spaces of constant curvature. (Received November 26, 1935.)

48. Professor Leonard Carlitz: *On some asymptotic formulas in the additive theory of numbers.*

In this paper asymptotic formulas are derived for such sums as $\sum\alpha_1(m_1)\cdots\alpha_\nu(m_\nu)$, the summation extending over all positive integers m_j and primes p_j such that $n=m_1+\cdots+m_\nu+p_1+\cdots+p_k$. The $\alpha_j(m)$ are arithmetic functions satisfying certain restrictions. (Received November 26, 1935.)

49. Professor L. R. Ford: *Complex rational fractions.*

Let the complex rational fraction $z=p/q$ be represented by a sphere of radius $1/2q\bar{q}$ lying in the half-space above the z-plane and tangent to the plane at p/q. The properties of these spheres are discussed. The following theorem is proved, the steps in the proof being suggested by the preceding picture: *If, after a certain point, the convergents p_n/q_n of the continued fraction whose partial quotients are $a_0, a_1, a_2, \cdots$ (a_n a complex integer) have the property $|q_{n+1}|>|q_n|$, then the fraction converges.* (Received November 26, 1935.)

50. Dr. W. T. Martin (National Research Fellow): *Special regions of regularity of functions of several complex variables.*

In the theory of functions of several complex variables various types of regions occur as convergence regions of series development of analytic functions, notable ones being *Reinhardt regions* of absolute convergence of power-series, *circular regions* of uniform convergence of diagonal series (the *invariant convergence regions*) and *Cartan* (p, q)*-regions* (p, q, positive integers) of uniform convergence of certain other series. In this paper we investigate the problem of analytically continuing a function $f(x, y)$ beyond its associated circular and Cartan regions. New regions B_{pq}, D_{pq}, and $S_{\sigma\mu}(\sigma, \mu$, positive numbers), of regularity of $f(x, y)$ are obtained (i) by a consideration of the absolute summability of Borel's integral means of certain series developments of $f(x, y)$; (ii) by the study of the behavior of a class of functional transforms of $f(x, y)$; and (iii) by a consideration of the behaviour of $f(x, y)$ along a family of curves through the origin. (Received November 26, 1935.)

51. Professor R. L. Moore: *Concerning essential continua of condensation.*

The continuum M is said to have property N if for every positive number e there exists a finite set H of non-degenerate subcontinua of M such that every subcontinuum of M of diameter more than e contains some continuum of H. The author has shown (see a forthcoming issue of the Rice Institute Pamphlet) that no compact continuum with property N has an essential continuum of condensation. It is now shown that if the compact and webless continuum M does not have property N there exists an upper semi-continuous collection of mutually exclusive continua filling up M and forming, with respect to its elements, a continuum with an essential continuum of condensation. In order that the compact and webless continuum M should have no essential continuum of condensation it is necessary and sufficient that every non-degenerate subcontinuum of M should contain uncountably many local separating points of M.

In order that such a continuum should have an essential continuum of condensation it is necessary and sufficient that there should exist a positive number d such that if G is an upper semi-continuous collection of mutually exclusive continua filling up M and forming, with respect to its elements, a continuum with no continuum of condensation, then some continuum of G is of diameter more than d. (Received November 26, 1935.)

52. Dr. Rufus Oldenburger: *Characteristic roots of direct products of n-way matrices.*

W. E. Roth has studied the characteristic roots of direct products of 2-way matrices. In the present paper characteristic roots of direct products P of two or more n-way matrices $A=(a_{ij\ldots k}), \cdots, D=(d_{pq\ldots s})$ are defined in terms of $P-\lambda I$, where I is a generalized Kronecker delta and λ is a complex variable. By means of theorems which express space determinants of P in terms of space determinants of $A, \cdots, D$ relations between the space ranks of $A, \cdots, D$ and the characteristic roots of P are obtained. In a paper to appear in a forthcoming issue of the Transactions the author has studied the non-singular linear transformations that leave I invariant. These transformations also leave the characteristic roots of P invariant. (Received November 26, 1935.)

53. Professor M. J. Weiss: *Fundamental systems of units in normal fields.*

Certain theorems on fundamental systems of units in cyclic fields given by C. G. Latimer (American Journal of Mathematics, vol. 56 (1934), pp. 69–74) are extended to fundamental systems of units in algebraic fields normal with respect to the rational field. This is done by considering the integral group ring, formed from the group of the field, as a ring of operators for the group of units in the field. (Received November 26, 1935.)

54. Professor R. L. Jeffery: *The equivalence of sequence integrals and non-absolutely convergent integrals.*

If the function $F(x)$ is continuous and is a non-absolutely convergent integral of the measurable function $f(x)$ on the interval (a, x), then there exists a sequence of summable functions $S_n(x)$ tending to $f(x)$ almost everywhere for which $TS(f, a, x)=\lim_{n\to\infty}\int_a^x S_n dx=F(x), 0\leqq x\leqq a$. The sequence S_n is such that $S_n=f$ on E_n, $S_n=0$ elsewhere, and mE_n tends to $b-a$. The present paper is concerned with the converse problem: If $f(x)$ is measurable, and if $TS(f, a, x)$ exists and is continuous, is $f(x)$ integrable in the generalized Denjoy sense? The answer is in the affirmative when for every perfect set P the points of non-summability of f over P are non-dense on P. (Received November 26, 1935.)

55. Dr. Max Zorn: *On elementary number theory.*

The existence of the primitive root mod p is known to be an instance of the algebraic theorem: The roots of unity in a field form a cyclic group, if their number is finite. In this paper it is shown that a simplification of the standard proof enables us also to derive abstractly a theorem of Ore, which contains a theorem of Schoenemann-Hensel about finite fields. The primitive root for odd

prime-powers is explained by means of the Fermat-Schur theorem. The Lemma of Gauss (and its extensions by Gauss himself, Artin-Schreier, and Carlitz) turns out to be a special case of a well-known theorem in group theory; this analysis gives immediately generalizations to higher fields. (Received November 26, 1935.)

56. Professor W. D. Baten: *The frequency distribution for the mean of n independent chance variables when each is subject to the law* $y_0 x^{p-1}(1-x)^{q-1}$.

This paper shows, by using certain results obtained from sampling theory and criterions for Pearson-type curves, which type represents the distribution of the mean of n independent chance variables when each is subject to the law $y_0 x^{p-1}(1-x)^{q-1}$. (Received November 26, 1935.)

57. Dr. R. H. Cameron: *Almost periodic properties of bounded solutions of linear differential equations with almost periodic coefficients.*

This paper deals with the differential equation $D(X)=AX$, where A is a given *a.p.* matrix function and $D(X)$ denotes the matrix obtained by differentiating each element of X. It is assumed that the real part of the sum of the diagonal elements of A has a bounded integral and that the equation has a bounded non-singular solution. Under these conditions it is shown that the equation has a non-singular solution in which the norm of each row is *a.p.* It is also shown that if the group of transformations of the solutions by limiting translations which leave A invariant is abelian, then the equation has a non-singular solution each element of which has an *a.p.* absolute value and an argument whose derivative is equal to the quotient of two *a.p.* functions except at points where the argument is indeterminate. (Received November 26, 1935.)

58. Dr. Robin Robinson: *On the differential line-geometry of space associated with the linear conformal transformations of the dual sphere.* Preliminary report.

A one-to-one correspondence between the lines of space and the points of the dual sphere is established by means of Study's "Uebertragungsprinzip." The dual sphere is then subjected to conformal transformations and the resulting invariants of space line-geometry are studied, with special reference to ruled surfaces and rectilinear congruences. This topic has already been treated by Knothe (Mathematische Zeitschrift, vol. 38 (1933–34), pp. 45–69), but the present writer uses a somewhat different method of attack, which follows closely the invariant methods recently used in this country by Graustein. It is also expected that the present paper, when ready for publication, will contain a number of new results. (Received November 26, 1935.)

59. Dr. S. B. Myers (National Research Fellow): *Isometries of closed 2-dimensional analytic Riemannian manifolds into themselves.*

It is easily shown that if T is an isometry of a closed analytic 2-dimensional

Riemannian manifold into itself, either T is of finite period or else the manifold admits a continuous group of isometries into itself. If T has a fixed point, this holds without the hypothesis of analyticity. Thus the study of isometries is naturally divided into the periodic case and the case of continuous groups. The latter case is treated by methods initiated by W. Rinow (Mathematische Annalen, vol. 107 (1932), pp. 95–112). In the periodic case, relations between the topology of the surface, the fixed points and fixed curves of T, and the period of T are studied. An important tool in this latter study is the "minimum point" locus previously studied by the author (Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 225–227 and Duke Mathematical Journal, vol. 1 (1935), pp. 376–391). (Received November 27, 1935.)

60. Mr. R. B. Kershner: *On singular Fourier-Stieltjes transforms.*

According to the Riemann-Lebesgue lemma, a necessary condition for the absolute continuity of a distribution function $\sigma(x)$ is that $\lim_{t\to\pm\infty} L(t, \sigma)=0$, where $L(t, \sigma)=\int_{-\infty}^{\infty} e^{itx}d\sigma(x)$. The present paper considers a class of distribution functions, $\sigma=\sigma_a(x)$, $0<a<1/2$, for which $L(t, \sigma_a)=\prod_{n=1}^{\infty}\cos(a^n t)$. These functions are not absolutely continuous; in fact, they are constant up to a set of measure zero. It is shown that if a is rational then $\lim L(t, \sigma_a)=0$ does or does not hold according as $1/a$ is not or is an integer. (Received November 27, 1935.)

61. Professor E. K. Haviland: *On the momentum problem for distribution functions in more than one dimension.*

It is shown that for the existence of a distribution function whose spectrum is contained in a given set C of the plane and which possesses the preassigned momenta c_{nm}, $(n, m=0, 1, 2, \cdots; c_{00}=1)$, it is necessary and sufficient that the matrix $\|c_{nm}\|$ be non-negative with respect to C, in the sense that to any polynomial $\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}x^n y^m$ non-negative for all (x, y) in C there corresponds the non-negative functional value $P_C=\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}c_{nm}$. This implies the well-known criteria for the various standard momentum problems as special cases. It turns out that this unified and more general treatment of all these momentum problems is no more complicated than the several individual treatments to be found in the literature for these special cases. (Received November 27, 1935.)

62. Professor J. L. Synge: *The dynamical theory of electrical commutator machinery.* Preliminary report.

Ingram (Philosophical Magazine, vol. 17 (1934), p. 844) and Kron (Journal of Mathematics and Physics, vol. 13 (1934), p. 103) have applied general dynamical methods to commutator machinery. In both cases, however, it is assumed that the problem of machine performance is distinct from the problem of commutation, and in fact that it is possible to write down equations of performance without paying attention to the currents in the windings short-

circuited by the brushes. In the work now reported the problem is investigated starting from the Lagrange-Maxwell equations and using as a basis of approximation the assumption that the number of commutator-segments is infinite. The investigation does not confirm the validity of the assumptions of Kron and Ingram. For a machine with shunt and interpoles in which the width of the brush does not exceed that of one commutator segment, explicit conditions for sparkless commutation are obtained, and also equations of performance.These involve the ratio of the widths of brush and commutator segment. (Received November 29, 1935.)

63. Dr. Max Zorn: *On subanalytic spaces.* I.

A Hausdorff space is said to be subanalytic if it has a defining neighborhood-system of a certain property, meaning roughly that the neighborhoods near a point either are small or contain neighborhoods of this point, some exceptions of a definite character permitted. Several variations of this concept are studied; each type yields the existence of arbitrarily small decompositions into pieces with different discreteness properties. "Arbitrarily small" is defined with arbitrary coverings by open sets, not by finite ones as usual. There are relations to generalized metrisability (that is, existence of a complete monotonic development of arbitrary length), generalized separability (that is, existence of a dense set which is the sum of a chain of discrete point sets), and generalized perfect separability. (Received November 26, 1935.)

64. Count Alfred Korzybski: *Extensionalization in mathematics, mathematical physics, and general education: general semantics.*

This paper deals with the following topics: *Experimental* character of extensionalization. Results with "normal" persons, defectives, and "mentally" ill. Definitions of "intension" and "extension." "Similarity of structure" as condition of predictability. Review of extensionalization in mathematics. Review of extensionalization in physics. The Einstein theory extensionalized physics. Its influence on the quantum theory. Semantic definition of "number" and mathematics. The elimination of the mystery "why" mathematics. Elimination of difficulties in teaching mathematics. Extensional general linguistic devices applicable to daily life and elementary education. The *experimental* wide evidence of unexpected possibility of application of *physico-mathematical methods* to daily life, applicable to small children, defectives and "mentally" ill, as well as in general education. The solution of problems of mathematical "infinity." The solution of the mathematical theory of "types." Conclusion: Once physico-mathematical methods are properly *evaluated* and formulated, (General Semantics) *experimental* evidence shows that these methods and orientations have a most beneficial influence on general human adjustment, developing automatically human "mentality," character, stability. *Preventive character* of extensionalization. The importance, scientific and human, *of more and more general theories* (E. H. Moore). (Received November 22, 1935.)

65. Mr. W. H. Ingram: *The Heaviside problem for the non-uniform transmission line.*

By the Heaviside problem is meant the problem of finding the current at any point in the line at any time subsequent to the sudden imposition of a constant electro-motive force at one or more points anywhere in the line. A solution of Bromwich-Heaviside type is here obtained, after recourse to the theory of linear integral equations, in which Fredholm's determinant plays a part analogous to $H(p)$ in Carson's integral equation. The result is generalizable to any passive non-self-modulating electrical system, continuous or discrete, and to the case of vibrating strings and air-columns. (Received November 29, 1935.)

66. Professor H. W. Raudenbush, Jr.: *On the Hilbert-Netto theorem for algebraic differential equations.*

The analogue obtained by J. F. Ritt (*Differential equations from the algebraic standpoint*, Colloquium Publications of this Society, vol. 14) for algebraic differential equations to the Hilbert-Netto theorem associates, for a given system, to each form having all the solutions of that system a minimum power which is a linear combination of the forms of the system and their derivatives with forms for coefficients. The analogy to the theory of algebraic equations suggests that the exponents of these powers are bounded. In this paper an example is given of a system for which these exponents are not bounded. (Received December 2, 1935.)

67. Professor Harris Hancock: *Minkowski's geometry of numbers.*

This paper gives a discussion of the history of Minkowski's geometry of numbers with certain applications. (Received December 4, 1935.)

68. Mr. W. E. Sewell: *A generalization of Markoff's theorem.*

Let $P_n(z)$ be a polynomial of degree n in z and let $|P_n(z)| < M$ on a point set whose complement is simply connected. Then $|P_n'(z)| < Kn^2$ on the set, K being a constant independent of n and z. This theorem is proved by a result stated in the writer's recent abstract, *Location of the Level Curves of Green's Function.* The present result is extended with suitable restrictions on the set to generalized derivatives and applied to the theory of approximation by polynomials. (Received December 15, 1935.)

69. Professor Marston Morse and Dr. A. E. Pitcher: *Invariants of closed extremals.*

The Poincaré rotation number has the following generalization. Let g be a closed extremal of length ω. Let x denote length on g. Let g^σ denote the closed extremal obtained by tracing g an integral number σ of times. Let ν_σ denote the number of conjugate points of $x=0$ on g^σ. The limit as σ increases without limit of ν_σ/σ exists. It is termed the frequency number of g. Let T_σ denote the index of g^σ. The limit as σ increases without limit of T_σ/σ exists and

is equal to the frequency number. The extremal g determines a matrix A of constants. If no characteristic root of A has the absolute value 1, the frequency number of g is an integer. The index of g is equal to ν_1 plus the order of concavity of g minus the count of secondary extremals of period ω vanishing at $x=0$. The number of conjugate points of $x=0$ preceding $x=\sigma\omega$ is equal to ν_1 plus $\nu_{\sigma-1}$ plus the index of repletion of g^σ minus the count of secondary extremals vanishing at $x=0$, $(\sigma-1)\omega$, and $\sigma\omega$. The index T_σ can be determined by a finite constructive process when A and ν_1 are given. (Received December 16, 1935.)

70. Professor J. F. Ritt: *Indeterminate expressions involving an arbitrary function and its derivatives.*

Let u represent an arbitrary analytic function of x and A and B two polynomials in u and a certain number of its derivatives with analytic functions for coefficients. The problem considered here is that of attributing a meaning to the quotient A/B when u is such as to cause A and B to vanish identically in x. In the author's Colloquium Lectures a prescription was laid down for the interpretation of such indeterminate expressions, on the basis of which it was conjectured that, when $A=B=0$, A/B represents either a single analytic function or all analytic functions. The present paper verifies this conjecture for the case in which A and B involve no derivative higher than the first derivative. (Received December 11, 1935.)

71. Professor H. S. White: *Cross-points of lines on a cubic surface.*

A cubic surface contains 27 straight lines, each meeting 10 others. Of these 135 cross-points, there are certain sets of 6 (no two on any of the 27 lines) which are coplanar. These have been mentioned in a paper of I. Schur, but apparently not elsewhere. By the aid of elliptic parameters in a variable plane section of the surface, a direct method is shown for testing whether any plane contains more than three such points. As a new item it is proved that no plane except those through some one or more of the 27 lines contains more than 6 cross-points, and that all those having 6 points form a single set. (Received Dec. 19, 1935.)

72. Professor Morris Marden: *A Grace-Heawood theorem for the critical points of Green's function.*

Concerning the critical points of Green's function, Professor Walsh has recently published a number of theorems suggested by certain results regarding the zeros of the derivative of a polynomial. (See, for example, American Mathematical Monthly, vol. 42 (1935), pp. 1–17) The main theorem of the present paper, an analogue of the Grace-Heawood theorem, is the following: Let R be an infinite region with a finite boundary B of which B_1 and B_2 are any two closed branches. Let C_1 (center α_1, radius r_1) and C_2 (center α_2, radius r_2) be any two circles which enclose B_1 and B_2 respectively and which have no interior points in common. Then, if no critical point of $G(x, y)$, Green's function for R and pole at infinity, lies in the closed exterior of hyperbola

$H(C_1, C_2)$ (foci α_1 and α_2, transverse axis r_1+r_2), at least one critical point of $G(x, y)$ lies in each interior of $H(C_1, C_2)$. (Received December 29, 1935.)

73. Professor W. E. Roth: *On the matrices AB and BA.*

The present note is primarily concerned with the proof of the theorem: *If A and B are $n \times n$ matrices with elements in the commutative field F, then the necessary and sufficient condition that AB and BA are similar in F is that a non-singular matrix T with elements in F exists such that AT and $T^{-1}B$ are commutative.* In case one of the matrices, say B, is non-singular, the matrix $T^{-1}B$ satisfies the above theorem for then $AT=AB$ and $T^{-1}B=I$ are commutative. The fact here noted that AB and BA are always similar matrices if either A or B is non-singular is known. (Wedderburn, *Lectures on matrices*, Colloquium Publications, vol. 17 (1934), p. 25.) The above theorem, however, takes care also of the case where both A and B are singular. The sufficiency proof of the theorem is trivial. (Received December 29, 1935.)

74. Dr. S. C. Kleene: *A note on recursive functions.*

The general recursive functions of natural numbers (under the Herbrand-Gödel definition) are the functions obtainable by repeated applications of the operation of forming, from a given function $\rho(x_1, \cdots, x_n, y)$ such that $\rho(x_1, \cdots, x_n, y)=0$ has solutions for every $x_1, \cdots, x_n$, the function (denoted by $\epsilon y[\rho(x_1, \cdots, x_n, y)=0]$) which the least solution is of $x_1, \cdots, x_n$, and of substitution, from the three particular functions $x+y$, $x \cdot y$, δ^x. (Received December 31, 1935.)

75. Professor Salomon Bochner: *Summation of general Fourier expansions on closed Lie groups.*

Recently the author investigated the summability of multiple Fourier series by so-called spherical means. The resulting criteria may be easily extended from ordinary multiple Fourier series to Fourier expansions on closed semi-simple Lie-groups. In particular the summability by spherical Riesz-means of a sufficiently high exponent depends solely on the local behavior of the given function, the smallest admissible exponent depending only on the dimension of the underlying group and not its specific nature. A detailed account will appear in the Annals of Mathematics. (Received January 1, 1936.)

76. Professor W. C. Graustein: *A new form of the four-vertex theorem.*

The "Vierscheitelsatz," in its usual form, states that an oval has at least four vertices, that is, that the curvature of an oval has at least four relative extrema. The new form says that an oval has at least four *primary* vertices. By a primary vertex is meant an extremum of the curvature which, if a maximum, is greater than the average curvature, and, if a minimum, is less than the average curvature. (Received January 2, 1936.)

77. Professor W. C. Graustein: *Applicability with preservation of both curvatures.*

The determination of conditions necessary and sufficient that there exist a surface applicable to a given surface with preservation of both the total and mean curvatures constitutes a problem of classical differential geometry which has received no little attention. In this paper, various new conditions, all in invariantive form, are found, and the map of a surface satisfying these conditions on a surface applicable to it in the manner described is shown to have many interesting geometrical properties. (Received January 2, 1936.)

78. Professor G. T. Whyburn: *Irreducible and arc-preserving transformations.*

If A is a compact continuum, a continuous transformation $T(A)=B$ will be called irreducible provided no proper subcontinuum of A maps onto all of B under T; T is said to be arc-preserving provided the image of every simple arc is either an arc or a single point; if A is locally connected, T is said to be A-set reversing provided the inverse of every point in B is either a single point or an A-set in the sense of the cyclic element theory. These types of transformations are investigated and relations between them are cleared up. If A is locally connected and T is A-set reversing, then for each true cyclic element E_b in B there exists a unique cyclic element E_a of A such that $T(E_a)=E_b$ and T is a homeomorphism on E_a. Also, if A is locally connected and T is irreducible and arc-preserving, T is a homeomorphism on each true cyclic element of A; thus in particular if A is cyclicly connected, T is a homeomorphism. Consequently every continuous irreducible transformation of one simple closed curve into another is a homeomorphism. Other particular cases, notably that in which B is a dendrite, are studied. (Received December 20, 1935.)

79. Professor G. T. Whyburn: *Completely alternating transformations.*

A continuous transformation $T(A)=B$ which is not (1-1) will be called completely alternating provided that if x, $y\epsilon B$ and x_1, $x_2\epsilon T^{-1}(x)$, then $T^{-1}(y)$ separates x_1 and x_2 in A. In this paper it is shown that if A is a compact continuum and $T(A)=B$ is completely alternating, then B is a simple closed curve and A is atriodic. Thus if A is locally connected, A is either an arc or a simple closed curve. If A is a simple closed curve, T is equivalent to the transformation $w=z^k$ on the circle $|z|=1$, where k is a positive integer, in the sense that if this second transformation is denoted by $W(A')=B'$, then one can write $T(A)=H_2WH_1(A)=B$, where $H_1(A)=A'$ and $H_2(B')=B$ are homeomorphisms. If A is a simple arc, T is equivalent in the same sense to the transformation $x=\cos(k+\frac{1}{2})t$, $y=\sin(k+\frac{1}{2})t$ on the interval $0\leqq t\leqq 2\pi$, where k is a positive integer. Finally, if it is agreed to call a transformation T as above completely componentwise alternating provided $T^{-1}(y)$ separates any two components of $T^{-1}(x)$ in A, then any such T can be factored into the form T_2T_1 where T_1 is monotone and T_2 is completely alternating. (Received December 20, 1935.)

80. Mr. Garrett Birkhoff: *Partially ordered sets and the inclusion relation.*

This is an expository article describing recent advances and applications of the abstract theory of ordering. The only new results are: (1) the remark that every "simply ordered" system is a distributive lattice, (2) the precise statement of the correspondence (called to the author's attention by L. R. Wilcox) between lattices and the "extensionally attainable properties" of E. H. Moore, (3) the proof that every "dimension function" maps the lattice on which it is defined homeomorphically onto a modular lattice, and every "weight function" onto a distributive lattice, and (4) the observation that inclusion can serve as the only primitive idea of combinatorial analysis situs. (Received December 28, 1935.)

81. Mr. Garrett Birkhoff: *Moore-Smith convergence in general topology.*

It is shown that the "convergence of systems of sets" described in Abstract 41-9-355 is equivalent in any Hausdorff space H to an extension of Moore-Smith convergence. It is proved that use of this notion eliminates the undesirable distinction between neighborhood closure and sequential closure. Finally, "completeness" is defined from a philosophical and logical viewpoint by three simple properties. (Received December 28, 1935.)