ON EXTENDING A HOMEOMORPHISM BETWEEN TWO SUBSETS OF SPHERES*

BY H. M. GEHMAN

In two papers previously published,† the author has determined conditions under which a homeomorphism, or continuous (1-1) correspondence, between two plane point sets of a certain type can be extended to a homeomorphism between their planes. The two types of point set which have been considered are (a) a continuous curve, and (b) a closed bounded set, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number. In a recent paper,‡ Adkisson has determined, for case (a), conditions under which a homeomorphism between two subsets of spheres can be extended to a homeomorphism between the spheres. The object of this paper is to generalize Adkisson’s results by proving a similar theorem for case (b). Finally it is shown how any theorem concerning the extension of a homeomorphism between plane sets yields a corresponding theorem for subsets of spheres, and conversely.

DEFINITION.§ An E-set is a closed proper subset of a sphere, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number.

THEOREM.|| Let M and M' be E-sets on the spheres S and S'
respectively, and let $T$ be a homeomorphism such that $T(M) = M'$. If $S - M$ and $S' - M'$ contain points $x$ and $x'$, respectively, such that $T$ preserves sides in the same sense in the planes $S - x$ and $S' - x'$, then $T$ can be extended to a homeomorphism $U$ between the spheres $S$ and $S'$. Conversely, if $T$ can be extended to a homeomorphism between the spheres $S$ and $S'$, then $T$ preserves sides in the same sense in the planes $S - x$ and $S' - x'$, where $x$ is any point of $S - M$, and $x' = U(x)$.

By Theorem 2 of Second paper, we know that $T$ can be extended to a homeomorphism $V$ between $S - x$ and $S' - x'$. Let us define a correspondence $U$ between $S$ and $S'$ as follows: $U(x) = x'$; for each point $y$ of $S - x$, $U(y) = V(y)$. This correspondence is evidently $(1-1)$. If $M$ is a subset of $S$, and a point $y$ of $S - x$ is a limit point of $M$, and hence of $M - Mx$, then, since $V$ is continuous, the point $U(y) = V(y)$ is a limit point of $U(M - Mx)$ and hence of $U(M)$. If $x$ is a limit point of $M$, then in the plane $S - x$ the set $M - Mx$ is unbounded. Hence $V(M - Mx)$ is unbounded in the plane $S' - x'$, and consequently the point $x' = U(x)$ of $S'$ is a limit point of $V(M - Mx) = U(M - Mx)$ and of $U(M)$. Hence the correspondence $U$ preserves limit points. In the same way it may be shown that $U^{-1}$ preserves limit points. Hence $U$ is the required homeomorphism. The converse part of the theorem is obvious.

Since a sphere minus a point is topologically equivalent to a plane, and since by the argument used above, a homeomorphism between two such planes can be extended to a homeomorphism between the spheres containing them, it follows that any theorem concerning the extension of a homeomorphism between subsets of planes yields a theorem concerning the extension of a homeomorphism between certain subsets of spheres.

Similarly a plane plus a point at infinity is topologically equivalent to a sphere. Any homeomorphism between two such spheres under which the points at infinity correspond, defines a homeomorphism between the two planes. Hence any theorem concerning the extension of a homeomorphism between proper subsets of spheres yields a corresponding theorem for certain subsets of planes.

The above remarks also hold true if we consider the extension
of a homeomorphism in the sense of Antoine.* From Theorem 2, p. 394, of the paper just cited, we can obtain a theorem for A-extending a homeomorphism between two subsets of spheres.

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A PROPERTY OF THE SOLUTIONS OF \( p^2 - du^2 = 4 \)

By Gordon Pall

Let \( p \) be any odd prime not dividing \( d \). The integral solutions \( t_i, \ u_i, \ (i = 0, \pm 1, \cdots) \),† of \( p^2 - du^2 = 4 \) have the following property.

Theorem. Let \( m+n = r+s \). Let \( v \) stand for \( t \) or \( u \). Then \( v_m + v_n \equiv v_r + v_s \; (\text{mod } p) \) if and only if the terms are congruent in pairs;‡ the same holds for each of

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\begin{align*}
v_m - v_n &\equiv v_r - v_s, \quad v_m + v_n \equiv -(v_r + v_s), \quad v_m - v_n \equiv -(v_r - v_s).
\end{align*}
\]

For if \( m+n \) is even and \( v = u \), we can write \( m = h+i, \ n = h-i, \ r = h+j, \ s = h-j, \) whence

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\begin{align*}
u_m + u_n &= u_{h+i}, \quad u_r + u_s = u_{h+j};
\end{align*}
\]

if \( u_h = 0 \), then \( u_m \equiv -u_n \); if \( t_i \equiv t_j \), known conditions for two \( u \)’s or \( t \)’s to be congruent show that \( u_m = u_r \) or \( u_s \). The remaining cases are similar. If \( m+n \) is odd, we transpose terms, and find with a little attention to parities (\( u_i = -u_{-i}, \ t_i = t_{-i} \)) one or other of the former cases.

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† For notations see, for example, Pall, Transactions of this Society, vol. 35 (1933), p. 501.
‡ That is, \( v_m \equiv -v_n, \ v_r, \) or \( v_s \).