exhibited by objects y_3 to y_n if and only if x_1, x_2, and y_3 to y_n exhibit ϕ. By $n-1$ such cumulative steps of interpretation, we find $(\cdots((\phi x_1)x_2)\cdots)x_{n-1}$ to be the "monadic relation" or attribute of being an object y_n such that the objects x_1 to x_{n-1} and y_n exhibit ϕ. The whole proposition applies this attribute to x_n and thus tells us that the objects x_1 to x_n exhibit ϕ.

HARVARD UNIVERSITY

REMARK ON A RECENT PAPER BY HOLLCROFT

BY A. R. WILLIAMS

1. *Introduction.* Among the characteristics of the general web of quadric hypersurfaces in S_r described by T. R. Hollcroft in a recent paper* is the number of lines on the jacobian surface of the web, that is, the number of hyperquadrics belonging to the web that have a line of vertices. This and more difficult questions are treated elegantly by associating the hyperquadrics of the web with the planes of a three-space. A direct algebraic treatment of the first mentioned problem may be of interest.

2. *Algebraic Formulation of the Problem. The Web of Conics.* For a quadric to have a line of vertices it is necessary and sufficient that all the first minors, but not all the second minors, of its discriminant vanish. This is three essential conditions. That is, in the linear system, or web, $\lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3+\lambda_4 f_4$, where the f's are linearly independent quadratic forms in any number of variables, a certain number have a line of vertices. While not strictly necessary it will make for clearness to begin with a web of conics. Let the discriminant be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

where the elements are linear homogeneous functions of λ_1, λ_2, λ_3, λ_4, and $a_{ij}=a_{ji}$. The three first minors in the first two rows represent quadrics which have a cubic k in common. Any two of them

* This Bulletin, vol. 41 (1935), p. 97.

intersect in this cubic and a line which meets it twice. The quadric $a_{12}a_{33}-a_{13}a_{32}=0$ meets the cubic k in 6 points which lie on the cubic κ, common to the three minors of the second and third columns, except those that lie on the line $a_{12}=a_{13}=0$ which is the residual intersection of $a_{12}a_{33}-a_{13}a_{32}=0$ with $a_{12}a_{23}-a_{13}a_{22}=0$. Now the cubic k and the line $a_{13}=a_{23}=0$ constitute the complete intersection of the quadrics $a_{12}a_{23}-a_{13}a_{22}=0$ and $a_{11}a_{23}-a_{13}a_{21}=0$. Hence where the line $a_{12}=a_{13}=0$ meets the second quadric it meets either the cubic k or the line $a_{13}=a_{23}=0$. But it meets the latter once, and hence k once. Hence the number of points common to k and κ, that is, the number of sets of values of the λ's for which the five distinct minors in the first two rows and the last two rows vanish, is $2\cdot 3-(2\cdot 1-1)=5$. This might have been inferred directly from the fact that they are cubics on the same quadric having as bisecants generators of different reguli. But the same method applies to the next case; and the intersections of two such cubics play the same role as the intersection of the two lines above. To finish the web of conics we note that of the 5 points just found one is $a_{21}=a_{22}=a_{23}=0$; and it does not cause the minor of a_{22} to vanish. Hence the number of conics of the web that have a line of vertices, and are therefore the squares of linear forms, is 4.

3. *The Web of Hyperquadrics in S_{n-1}.* In the general case, the discriminant of a quadratic form in n variables is a symmetric determinant of n rows and columns. In our problem the elements are homogeneous linear functions of the four λ's. In these variables the n first minors of the first $n-1$ rows represent surfaces of order $n-1$ which have in common a curve C of order $n(n-1)/2$. Any two of these surfaces intersect in the common curve and a residual curve R of order $(n-1)(n-2)/2$. To find in how many points R meets C we note that the points in which R meets a third first minor surface taken from the same $n-1$ rows satisfy the n first minors of these rows, except those points for which vanish all determinants of the matrix of $n-1$ rows and $n-3$ columns common to the three first minors in question. Therefore the number of points in which R meets C is given by $(1/2)(n-1)^2(n-2)-(1/6)(n-1)(n-2)(n-3)=n(n-1)(n-2)/3$. Any two residual curves R on the same first minor surface intersect in $(n-1)(n-2)(n-3)/6$ points.

Likewise the $n-1$ first minor surfaces taken from the last $n-1$ columns (rows) have in common a curve Γ which with C lies on $A_{n1}=0$, where A_{n1} is the minor of a_{n1}, and is in the upper right-hand corner of the discriminant. Let the number of intersections of C and Γ be $f(n)$. Just as above, the surface $A_{n-1,1}=0$ meets C in $(n/2)(n-1)^2$ points which lie on Γ except those which lie on γ, where γ, of order $(n-1)(n-2)/2$, is the residual intersection of $A_{n1}=0$ and $A_{n-1,1}=0$, and is the locus of points for which vanish all determinants of the matrix of $n-2$ rows and $n-1$ columns common to A_{n1} and $A_{n-1,1}$. The complete intersections of $A_{n1}=0$ and $A_{n2}=0$ is C and a curve c of the same order as γ, corresponding to the matrix of $n-1$ rows and $n-2$ columns common to A_{n1} and A_{n2}. Where γ meets $A_{n2}=0$ it meets either C or c. But γ and c lie on a surface of order $n-2$, and the number of their intersections is $f(n-1)$. Hence we have $f(n)=(n/2)(n-1)^2-[(1/2)(n-1)^2(n-2)-f(n-1)]$; or $f(n)-f(n-1)=(n-1)^2$. Hence $f(n)=1^2+2^2+\cdots+(n-1)^2=(n-1)(n)(2n-1)/6$. This is the number of sets of λ's for which vanish the $2n-1$ distinct first minors corresponding to the elements of any two rows or columns, in our case the first and last rows. This gives the key to the main problem. We have merely to subtract from the last result the number of points common to all the second minors made from the intervening rows, that is, the second to the $(n-1)$st inclusive. Hence the number of quadrics of the web that have a line of vertices is

$$\frac{(n-1)(n)(2n-1)}{6}-\frac{n(n-1)(n-2)}{6}=\frac{(n+1)(n)(n-1)}{6}.$$

Putting $n=r+1$ we have the expression given by Hollcroft. It may be remarked that if the coefficients are linear functions of five homogeneous variables connected by a relation of order m, the number of quadrics of the non-linear web that have a line of vertices is $m(n+1)(n)(n-1)/6$.

4. *Quadratic Forms Which are the Products of Two Linear Forms or the Squares of One Linear Form.* For a quadric to have a plane of vertices all the second minors of its discriminant must vanish. This is six essential conditions. The above method results in algebraic difficulties even in linear systems. But the two

extreme cases when the form is the product of two distinct linear forms, or the square of one, can be dealt with geometrically. For a quadratic form to be the product of two distinct linear forms is $(n-1)(n-2)/2$ essential conditions, n being the number of homogeneous variables. A sufficiently general linear system having that many degrees of freedom is furnished by the quadrics having $2n-2$ simple points in common. These may be separated into two sets of $n-1$, each determining a $(n-2)$-space in $(2n-2)!/2[(n-1)!]^2$ ways. For $n=3$ we have a pencil of conics containing 3 linear pairs. For $n=4$ we have the web of quadrics through 6 points and containing 10 pairs of planes, and so on. More interesting is the case when the form is a perfect square. For this to be true is $n(n-1)/2$ essential conditions. Here we can use the notion of apolarity. If a quadric in point coordinates is a square, a quadratic envelope apolar to it is tangent to it. Apolar to $n(n-1)/2+1$ quadratic forms in point coordinates and to the linear system determined by them, is a linear system of ∞^{n-2} envelopes based on $n-1$ independent ones. The number of perfect squares in the first family is the number of hyperplanes common to these $n-1$ envelopes, which is 2^{n-1}. For $n=3$ we have the 4 conics of a web that are squares. For $n=4$ we have the 8 quadrics in a linear system of ∞^6 which are squares.

THE UNIVERSITY OF CALIFORNIA