TRANSFORMS OF FUCHSIAN GROUPS

BY P. K. REES

This paper gives four theorems concerning the relative sizes of the isometric circles of the transformations, \( T(z) = (az + c)/(cz + \bar{a}) \), of a Fuchsian group and those of the transforms, \( S(z) = GTG^{-1}(z) = (A\bar{z} + \bar{C})/(C\bar{z} + \bar{A}) \), of \( T \) in which \( G(z) = (\alpha z + \nu)/(\nu z + \bar{\alpha}) \) is considered as fixed and \( T \) any transformation of the Fuchsian group.

THEOREM 1. The necessary and sufficient condition that the radii, \( r_s \) and \( r_t \), of the isometric circles of \( S \) and \( T \) be equal is that the midpoint, \((a - \bar{a})/(2c) - m\), of the line segment joining the centers, \( g_t \) and \( g'_t \), of the isometric circles, \( I_t \) and \( I'_t \), of \( T \) and \( T^{-1} \) be on the circle \( Q_\delta(z) \) with the origin and the center, \( g = -\bar{\alpha}/\nu \), of the isometric circle of \( G \) as opposite ends of a diameter or on the circle \( Q'_\delta(z) \) with the origin and \( 1/\bar{\delta} \) as opposite ends of a diameter.

PROOF. The equations of \( Q_\delta(z) \) and \( Q'_\delta(z) \) are
\[
Q_\delta(z) = 2\bar{\nu}zz + \alpha \nu z + \bar{\alpha} \nu \bar{z} = 0, \quad Q'_\delta(z) = 2\alpha \bar{\nu}zz + \alpha \nu z + \bar{\alpha} \nu \bar{z} = 0.
\]
If \( z \) lies on either \( Q_\delta \) or \( Q'_\delta \), then \( Q_\delta(z)Q'_\delta(z) = 0 \). But
\[
\frac{1}{r_s^2} - \frac{1}{r_t^2} = -\frac{(a - \bar{a})(-\alpha \nu \bar{c} + \bar{\alpha} \nu c) - \alpha \bar{\alpha} \nu \nu[\alpha - \bar{a})^2 - 2c\bar{c}]}{(\alpha \nu \bar{c})^2 - (\bar{\alpha} \nu c)^2},
\]
which vanishes if and only if \( r_s = r_t \). Multiplying (1) equated to zero by \( -(a - \bar{a})(2c\bar{c}) \) and replacing \( (a - \bar{a})/(2c) \) by \( m \), we have \( Q_\delta(m)Q'_\delta(m) = 0 \).

THEOREM 2a. The necessary and sufficient condition that \( r_s < r_t \) \((r_s > r_t) \) is that \( z = m \) substituted in the expression for \( Q_\delta Q'_\delta \) makes that expression positive (negative).

PROOF. \( r_s \leq r_t \) according as \( 1/r_s^2 - 1/r_t^2 \geq 0 \). Furthermore
\[
Q_\delta Q'_\delta = -\left(\frac{1}{r_s^2} - \frac{1}{r_t^2}\right)(a - \bar{a})^2 / 4c^2\bar{c}^2.
\]

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Therefore \( Q_5Q_6 \leq 0 \) according as \( 1/r_s^2 - 1/r_t^2 \leq 0 \), that is, according as \( r_s \geq r_t \).

**Theorem 2b.** The necessary and sufficient condition that \( r_s < r_t \) is that \( m \) be outside both \( Q_5 \) and \( Q_6 \) or inside both; the necessary and sufficient condition that \( r_s > r_t \) is that \( m \) be inside \( Q_5 \) or \( Q_6 \) and outside the other.

**Proof.** The expressions for each \( Q_5 \) and \( Q_6 \) are negative (positive) according as \( m \) is inside (outside) the circle. Theorem 2b follows from this and Theorem 2a.

**Remark 1.** The diameter of \( Q_5 \) is equal to \( |g| = |\bar{a}/\nu| \). This can be made as large as one may wish by choosing \( |\nu| \) sufficiently near zero. Furthermore, the radius of \( Q_6 \) is the reciprocal of that of \( Q_5 \). Hence, by choosing \( G \) with \( |\nu| \) sufficiently near zero, one can make the region inside \( Q_5 \) or \( Q_6 \) and outside the other as nearly a half-plane as desired. Therefore, for \( g \) sufficiently large, those transformations of the group \( T \) with \( m \) in approximately one half-plane (the one \( g \) is in) have their isometric circles increased in magnitude by transforming by \( G \) whereas those with \( m \) in the other approximate half-plane have \( r_s < r_t \).

Furthermore by choosing \( |g| \) sufficiently near to unity one can make the region inside \( Q_5 \) or \( Q_6 \) and outside the other as small as he may wish. Thus the transformations with \( m \) in as nearly the entire plane as desired have their isometric circles decreased in magnitude by transforming by \( G \).

**Theorem 3.** The necessary and sufficient condition that \( r_s = r_t/k \), \( k \) a non-negative real number, is that \( m \) lie on the locus

\[
(2) \quad (2a\bar{a}\bar{v}v \bar{z} + a\bar{v}v \bar{z} + a\bar{a}\bar{v}z)(2a\bar{a}\bar{v}v \bar{z} + a\bar{v}v \bar{z} + a^2\bar{v}z) = k^2\bar{z}\bar{z}a\bar{a}\bar{v}v.
\]

**Proof.** From the definitions of \( r_s \) and \( r_t \) and from the equation \( r_s = r_t/k \), we have \( (r_t/r_s)^2 = (C\bar{C})/(c\bar{c}) = k^2 \). Replacing \( C\bar{C} \) by its value in terms of the coefficients of \( T \) and \( G \) and then replacing \((a-\bar{a})/(2c)\) by \( m \), we have (2), since \( c/\bar{c} = -\bar{m}/m \).

**Remark 2.** The number \( k \) is not determined by (2) for a real, since then \( m = 0 \). However, \( m \) is on both \( Q_5 \) and \( Q_6 \) for \( m = 0 \), and therefore, by Theorem 1, \( k = 1 \).
Corollary 1. The absolute minimum value of \( k \) is zero; this value is taken on if the midpoint of the line segments \((g_t, g'_t)\) and \((g, 1/\bar{g})\) coincide and is possible only for \( T \) an elliptic transformation.

Proof. Substituting \( m = -\frac{(\alpha \bar{\alpha} + \nu \bar{\nu})}{(2\alpha \nu)} \) into (2), we see that \( k = 0 \) if \( (a - \bar{a})/(2c) = -\frac{(\alpha \bar{\alpha} + \nu \bar{\nu})}{(2\alpha \nu)} \). Furthermore, we have \( Q_0\left[-\frac{(\alpha \bar{\alpha} + \nu \bar{\nu})}{(2\alpha \nu)}\right] > 0 \) for all \( G \) and all \( T \) of Fuchsian type, whereas \( Q_0\left[\frac{(a - \bar{a})}{(2c)}\right] > 0 \) for \( T \) elliptic only.

Remark 3. Changing (2) to trigonometric form, one finds the discriminant of the resulting quadratic in \( \rho \) to be

\[
 f(k) = 4(\alpha \nu e^{i\theta} + \bar{\alpha} \bar{\nu} e^{-i\theta})^2 - 16\alpha \bar{\alpha} \nu \bar{\nu} (1 - k^2).
\]

This is a perfect square if and only if \( k = 1 \) or 0; hence (2) is factorable rationally in terms of the coefficients of \( G \) in these two cases and only in them. The factors for \( k = 1 \) are \( Q_0 \) and \( Q_0 \) of Theorem 1, and for \( k = 0 \) they are immediate from (2).

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THE EQUATION \( 2^x - 3^y = d^* \)

BY AARON HERSCHELF

1. Introduction. According to Dickson’s History of the Theory of Numbers,† Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that \( 3^m \pm 1 \neq 2^n \) if \( m > 2 \), by showing that \( 3^m \pm 1 \) has an odd prime factor. The problem had been proposed to him by Philipp von Vitry in the following form: All powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. In 1923 an elegant short proof by Philip Franklin appeared in the American Mathematical Monthly.‡

In 1918 G. Polya§ published a very general theorem which, as was later pointed out by S. Sivasankaranarayana Pillai,‖ proved as special cases that the equations

* Presented to the Society, October 26, 1935.
‡ Vol. 30 (1923), p. 81, problem 2927.