SHORTER NOTICES

Leçons sur la Théorie Mathématique de la Lutte pour la Vie. By Vito Volterra. Rédigées par M. Brelot. (Cahiers Scientifiques, vol. 7.) Paris, Gauthier-Villars, 1931. vi+214 pp.

A biological system is defined as a finite number of species of animals living in a common milieu. The animals may, for example, be fishes off Newfoundland, bacteria in a host, game animals in a forest. The animals compete for food, and some prey on others. Let $N_i(t)$, $(i=1, \cdots, n)$, be the number of animals of the ith species alive at time t. Professor Volterra has undertaken the problem of investigating the functions $\{N_i(t)\}$.

It is frankly stated that "although we shall use biological *terminology*, the book will actually be a *purely mathematical* study of certain positive solutions of integro-differential equations." The functions $\{N_i(t)\}$ are considered as differentiable functions, and the results are not applied to a specific numerical example, although the qualitative results are presumably applicable to the many concrete biological systems mentioned.

The book is divided into two sections. In the first, the development of a biological system is supposed completely determined by its state at any given time. This leads to systems of differential equations, such as

$$\frac{dN_r}{dt} = \left(\epsilon_r - \sum_1^n p_{rs}N_s\right) N_r, \qquad (r = 1, \cdots, n), \tag{1}$$

(where ϵ_r, p_{rs} are independent of t). The second part introduces the effect of the past history of the biological system. This leads in a natural way to integro-differential equations, such as, $(n=2)$,

$$\begin{aligned} \frac{dN_1}{dt} &= \left[\epsilon_1 - \gamma_1 N_2(t) - \int_0^\infty F_1(\tau)N_2(t-\tau)d\tau\right] N_1(t), \\ \frac{dN_2}{dt} &= \left[-\epsilon_2 + \gamma_2 N_1(t) + \int_0^\infty F_2(\tau)N_1(t-\tau)d\tau\right] N_2(t), \end{aligned} \tag{2}$$

where $\epsilon_i \geqq 0$, $\gamma_i \geqq 0$, $F_i \geqq 0$. Particular emphasis is laid on solutions in the neighborhood of a stationary solution (one in which $N_i(t)$ is independent of t) and on the effect of perturbations. Many interesting particular cases are examined in great detail.

Several times in the book the expression "infinitely improbable" is met—referring to a special case which the author wishes to exclude. This expression is frequently used in probability to replace "true with probability 0," perhaps as a relic of the days when no one knew anything very precise about either probability or infinity. It should not be used to replace a real justification of an exclusion of a particular case where no probability is explicitly involved. Thus in investigating solutions of (1) (with n even and with certain restrictions on the constants concerned) in the neighborhood of a stationary solution, it is found that $N_i(t)$ is approximately a sum of $n/2$ sinusoidal fluctuations. To draw the conclusion that $N_i(t)$ is not periodic it is necessary to suppose that the corresponding $n/2$ periods are incommensurable. To say that commen-

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surability is infinitely improbable and that therefore $N_i(t)$ is not periodic in practice implies the existence of a criterion of probability for the $n/2$ periods, although such a criterion would necessarily be less elementary than the criterion of periodicity of $N_i(t)$—which is presumably estimated directly from the biological system.*

This criticism is on something irrelevant to the real purpose of the book, as stated by the author in the sentence quoted above. The mathematical content of the book, a detailed analysis of certain differential and integro-differential equations, should be valuable both to the mathematician and to the statistician interested in the qualitative and quantitative study of the development of a biological system.

J. L. Doob

Interpolatory Function Theory. By J. M. Whittaker. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 33.) Cambridge University Press. New York, Macmillan, 1935. vii+107 pp.

This interesting tract by the son of E. T. Whittaker is devoted to the following problem. Let $\Pi_i f(0)$ denote the differential operator

$$\Pi_i f(0) = \sum_{j=0}^{\infty} \frac{\pi_{ij}}{j!} f^{(j)}(0).$$

What analytic functions $f(z)$ are uniquely determined by the values of $\Pi_i f(0)$, $i=0, 1, 2, 3, \cdots$? This problem contains a large number of important special cases such as expansion in Taylor's series, interpolation at positive integers or at lattice points, and Bernoullian series. The fundamental concept is the notion of basic sets. A set of polynomials

$$p_i(z) = \sum_{j=0}^{\infty} p_{ij} z^j$$

is said to be *basic* if every polynomial is a linear combination of a finite number of polynomials from the set. A necessary and sufficient condition that the set be basic is that the matrix $P = \| p_{ij} \|$ have a row-finite inverse. A set of operators is basic if the set of associated functions

$$p_i(z) = \sum_{j=0}^{\infty} \pi_{ji} z^j$$

is a basic set of polynomials, that is, if the transpose of the matrix $\Pi = \| \pi_{ij} \|$ is row-finite and has a row-finite inverse. The series

$$\sum_{i=0}^{\infty} \Pi_i f(0) p_i(z)$$

* There seems to be a general principle in many investigations that the constants which occur in it have a continuous probability distribution, so that it is "infinitely improbable" that these constants have any preassigned set of values. As in the case just discussed, this unjustified assumption may be almost precisely what was to be proved.

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