THE FORM $wx+xy+yz+zu$

BY E. T. BELL

1. *Introduction.* In the usual notation,

$$N \equiv N[n = wx + xy + yz + zu; \quad w, x, z, u > 0; \quad y \geqq 0]$$

denotes the number of sets (w, x, y, z, u) of integers, subject to the conditions indicated, satisfying the stated equation in which n is an arbitrary constant integer >0. Let $\zeta_r(n)$ denote the sum of the rth powers of all the divisors of n, so that $\zeta_0(n)$ is the number of divisors. Then

$$N = \zeta_2(n) - n\zeta_0(n). \tag{1}$$

This curious result is the only one of the numerous theorems on quadratic forms stated by Liouville for which (apparently) no proof has been published.*

We shall first show that (1) follows from

$$2N' = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n), \tag{2}$$

$$N' \equiv N'[n = wx + xy + yz + zu + ux; \quad w, x, y, z > 0; \quad u \geqq 0],$$

and then prove (2). Another similar result is stated in §5.

2. *Equivalence of* (1) *and* (2). The form in N' may be written

$$yz + (z + x)u + x(w + y);$$

and hence, by the conditions on the variables, $w+y \equiv y' > y$. Thus (2) is equivalent to

$$\begin{aligned} &\zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n) \\ &\quad = 2N'[n = yz + (z+x)u + xw; \; x, y, z, w > 0; \; u \geqq 0; \; w > y]. \end{aligned} \tag{3}$$

Applying the substitution $(xz)(yw)$ to the last we see that (3) holds also when the condition $w > y$ is replaced by $w < y$.

Consider now the remaining possibility, $w = y$. The equation becomes

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* J. Liouville, Comptes Rendus, Paris, vol. 62 (1866), p. 714; also, Journal de Mathématiques, (2), vol. 12 (1867), pp. 47–48. Noted in Dickson's *History*, vol. 3, p. 237. Liouville points out why the theorem is unusual.

$$(4) \qquad n = (x + z)(y + u); \qquad x, y, z > 0; \qquad u \geqq 0.$$

Hence if $n = d\delta$ is any resolution of n into a pair of positive divisors, the number of solutions of (4) for a fixed (d, δ) is

$$N[d = x + z;\ x, z > 0] \times N[\delta = y + u;\ y > 0;\ u \geqq 0],$$

that is, $(d-1)\delta$; and therefore the total number of solutions of (4) is $\sum(d-1)\delta$, the sum referring to all pairs (d, δ). Thus (4) has precisely $n\zeta_0(n) - \zeta_1(n)$ solutions.

But all the solutions of

$$(5) \qquad n = yz + (z + x)u + xw; \qquad x, y, z, w > 0; \qquad u \geqq 0$$

are exhausted by the three mutually exclusive sets in which $w > y$, $w < y$, $w = y$, respectively, and the number of solutions in each of these has just been determined. The total number of solutions of (5) is thus

$$\zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n) + n\zeta_0(n) - \zeta_1(n),$$

or $\zeta_2(n) - n\zeta_0(n)$. Hence (1) follows from (2). Conversely, by reversing the steps, (2) follows from (1), so that (1), (2) are equivalent.

3. *Dependence of* (2) *on an Auxiliary Relation* (6). Let $\phi(u, v)$ be finite and single-valued for all integer values of the variables u, v, and beyond the condition $\phi(u, v) = -\phi(v, u)$ for integer values of u, v, let $\phi(u, v)$ be arbitrary. Then

$$(6) \qquad \sum \phi(w + y, z) = \sum\left[\sum_{r=1}^{\delta-1} \phi(r, d)\right],$$

$\sum$ on the left referring to all (w, y, z), that on the right to all (d, δ), from

$$(7) \qquad d\delta = n = wx + xy + yz; \qquad d, \delta, w, x, y, z > 0,$$

in which all the letters denote integers and n is constant. Assuming this for the moment, we shall prove (2).

The form in (7) is invariant under the substitution $(xy)(zw)$. Hence $\sum w = \sum z$, the sums extending over all solutions of (7). Taking $\phi(u, v) \equiv u - v$ in (6), we get

$$(8) \qquad 2\sum y = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n),$$

the left member of which is

$$2\sum N[y = y_1 + y_2; \quad y_1 > 0; \quad y_2 \geqq 0].$$

It follows that

$$2N[n = wx + (x+z)(y_1 + y_2); \quad w, x, z, y_1 > 0; \quad y_2 \geqq 0]$$

is given by the right member of (8). By a change of notation for the variables this result is (2).

4. *Proof of* (6). We now prove (6). The functions $h(u), f(u, v)$ are single-valued and finite for integer values of the variables, and beyond the conditions (for integer values of u, or of u, v)

$$h(u) = h(-u), \qquad f(u, v) = f(-u, v) = f(u, -v),$$

are arbitrary. Hence in a theorem concerning $f(u, v)$ we may replace $f(u, v) - f(v, u)$ by $\phi(h(u), h(v))$, where ϕ is as in §3. For f we have the identity*

$$\sum [f(d_1 + d_2, \delta_1 - \delta_2) - f(\delta_1 - \delta_2, d_1 + d_2)]$$
$$= \sum\left[(d-1)\{f(d,0) - f(0,d)\} + 2\sum_{r=1}^{\delta-1} \{f(r,d) - f(d,r)\}\right],$$

the sum on the right referring to all (d, δ), that on the left to all $(d_1, \delta_1, d_2, \delta_2)$, such that

$$d\delta = n = d_1\delta_1 + d_2\delta_2, \tag{9}$$

in which all letters denote integers >0 and n is constant. Hence

$$\sum \phi(h(d_1 + d_2), h(\delta_1 - \delta_2))$$
$$= \sum\left[(d-1)\phi(h(d), h(0)) + 2\sum_{r=1}^{\delta-1} \phi(h(r), h(d))\right]. \tag{10}$$

In (10) take $h(u) \equiv |u|$. Then

$$\sum \phi(d_1 + d_2, |\delta_1 - \delta_2|)$$
$$= \sum\left[(d-1)\phi(d,0) + 2\sum_{r=1}^{\delta-1} \phi(r,d)\right]. \tag{11}$$

* Equivalent to one stated by Liouville, Journal de Mathématiques, (2), vol. 3 (1858), pp. 282–284. The first proof, by elementary means, was given by T. Pepin, ibid., (4), vol. 4 (1888), pp. 84–92; I showed that the identity is equivalent to one between doubly periodic functions of the first and second kinds (Transactions of this Society, vol. 22 (1921), p. 215).

According as $\delta_1-\delta_2>0$, $\delta_1-\delta_2<0$, $\delta_1-\delta_2=0$ we have

$$\delta_1 = \delta_2 + \delta_1', \; \delta_1' > 0, \qquad n = d_1(\delta_2 + \delta_1') + d_2\delta_2;$$
$$\delta_2 = \delta_1 + \delta_2', \; \delta_2' > 0, \qquad n = d_1\delta_1 + d_2(\delta_1 + \delta_2');$$
$$n = \delta_1(d_1+d_2).$$

The third of these contributes $\sum(d-1)\phi(d, 0)$, summed over all divisors d of n, to the left of (11). The forms in the first two are equivalent under the substitution $(d_1d_2)(\delta_1\delta_2)(\delta_1'\delta_2')$; for the first, $|\delta_1-\delta_2|=\delta_1'$, for the second $|\delta_1-\delta_2|=\delta_2'$. Hence, by the equivalence just noted, these two together contribute $2\sum\phi(d_1+d_2, \delta_1')$ to the left of (11). Substituting these results into (9), (11), and changing the notation,

$$(d_2, \delta_2, d_1, \delta_1') = (w, x, y, z),$$

we get (6).

5. *Another Similar Result.* Other choices of ϕ in (6) give theorems on numbers of representations. From the results already given it is easily seen that

$$N[n = wx + xy + yz + zu + ux;\ x, y > 0;\ u, z, w \geqq 0] = \zeta_2(n).$$

To prove this we require

$$2N[n = x(w + y + u); \quad x, y, w > 0; \quad u \geqq 0] = \zeta_2(n) - \zeta_1(n),$$

which follows at once on noting that $x=d$, $w+y+u=\delta$, where $n=d\delta$, and that

$$N[\delta = w + y + u; \quad w, y > 0; \quad u \geqq 0]$$

is the coefficient of q^δ in the expansion of $q^2(1-q)^{-3}$, and hence is $\delta(\delta-1)/2$.

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