ON THE MAGNITUDE OF THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

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Until very recently all the results of the investigations into the magnitude of the coefficients of the cyclotomic polynomial

\[ Q_n(x) = \prod_{d|n} (1 - x^{n/d})^{\mu(d)} \]

tended to show that these coefficients are very small indeed. In fact for \( n < 105 \) all the coefficients are \( \pm 1 \), and \( 0 \), and for \( n < 385 \) they do not exceed 2 in absolute value.

In 1883 Migotti* showed that the coefficients of \( Q_n(x) \) are all \( \pm 1 \) or 0 for \( n \) a product of two primes, but noted that the coefficient of \( x^7 \) in \( Q_{105}(x) \) is \(-2\). In 1895 Bang† proved that no coefficient of \( Q_n(x) \) for \( n = pq \), \( (p < q \), odd primes), exceeds \( p - 1 \).

Nothing further was done on the problem until 1931, when I. Schur gave a very ingenious proof of the following theorem.

**SCHUR’S THEOREM.** There exist cyclotomic polynomials with coefficients arbitrarily large in absolute value.

As this proof has not been published, it is given below.‡

**Proof.** Let \( n = p_1p_2 \cdots p_t \), where \( t \) is odd and \( p_1 < p_2 < \cdots < p_t \) are odd primes such that§ \( p_1 + p_2 > p_t \). To prove the theorem it is sufficient to show that the coefficient of \( x^{pt} \) in \( Q_n(x) \) is \( 1 - t \). This can be done by taking \( Q_n(x) \) modulo \( x^{pt+1} \). We then get

\[ Q_n(x) \equiv \prod_{i=1}^{t} (1 - x^{p_i})/(1 - x) \equiv (1 + x + \cdots + x^{p_t-1})(1 - x^{p_1})(1 - x^{p_2}) \cdots (1 - x^{p_t-1}) \equiv (1 + x + \cdots + x^{p_t-1})(1 - x^{p_1} - x^{p_2} - \cdots - x^{p_t-1}) \pmod{x^{pt+1}}. \]

* Sitzungsberichte, Akademie der Wissenschaften, Wien. (Math), (2), vol. 87 (1883), pp. 7–14.
† Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.
‡ This proof is essentially the one given by Schur in a letter to Landau.
§ Such a set of primes exists for every \( t \).
Collecting the coefficient of $x^{pt}$ in this last expression we see that it is precisely $-(t-1)$, so that as $t$ increases we can exhibit arbitrarily large negative coefficients of the cyclotomic polynomials, which proves the theorem.

The question now remains as to the boundedness of the coefficients of $Q_n(x)$ for a fixed $t$. We have already seen that for $t=1$ and 2 these coefficients are actually bounded. The case $t=3$ was discussed by Bungers* who proved the following theorem.

**Bungers' Theorem.** As $n$ runs over all products of three distinct primes, the cyclotomic polynomials $Q_n(x)$ contain arbitrarily large coefficients, provided there exist infinitely many prime pairs.

His proof depends on choosing three primes, two of which differ by 2, and in exhibiting a coefficient of $Q_{pqr}(x)$ equal to $(p+1)/2$. It is the purpose of this note to modify Bungers' proof so as to eliminate the unproved assumption of the existence of infinitely many prime pairs.

Let $n = pqr$, where $q = kp + 2$, and $r = (mpq - 1)/2$. For a given $p$ such primes $q$ and $r$ can always be found by Dirichlet's Theorem. We proceed to show that the coefficient of $x^h$, where $h = (p-3)(qr+1)/2$ is $(p-1)/2$ and hence can be made arbitrarily large with $p$. From (1) with $n = pqr$, we have

$$Q_{pqr}(x) = \frac{(x^{pqr} - 1)(x^p - 1)(x^q - 1)(x^r - 1)}{(x - 1)(x^{pq} - 1)(x^{pr} - 1)(x^{qr} - 1)}$$

$$= (1 + x + \cdots + x^{pqr})(1 - x^q - x^r + x^{q+r})$$

$$\cdot \sum x^{vqr+\lambda pr+\mu pq} \pmod{x^{pqr}}.$$ 

Since we are interested in the coefficient of $x^h$, the summation indices $v, \lambda, \mu$, satisfy the following inequalities:

$$vqr \leq h, \quad \lambda pr \leq h, \quad \mu pq \leq h.$$ 

We now consider the diophantine equation

$$vqr + \lambda pr + \mu pq + \omega + eq + \eta r = (p-3)(qr+1)/2 = h,$$

where $\omega < p$, and $\epsilon = 0$ or 1, $\eta = 0$ or 1.

The coefficient of $x^h$ is now given by the number of solutions of (3) with $\epsilon = \eta$ minus the number of solutions of (3) with $\epsilon \neq \eta$.

Taking (3) modulo $p$, $q$, and $r$ we have, since $qr \equiv -1 \pmod{p}$,
\[
\begin{align*}
vqr + \omega + eq + \eta r & \equiv 0 \pmod{p}, \\
\lambda pr + \omega + eq & \equiv (p - 3)/2 \pmod{q}, \\
\mu pq + \omega + eq & \equiv (p - 3)/2 \pmod{r}.
\end{align*}
\]

Multiplying the last two congruences by $k$ and $m$, respectively, and remembering that $kpr \equiv 1 \pmod{q}$, while $mpq \equiv 1 \pmod{r}$, also that $q \equiv 2 \pmod{p}$, and $r \equiv -1/2 \pmod{pq}$, we get
\[
\begin{align*}
\omega & \equiv v - 2e + \eta/2 \pmod{p}, \\
\lambda & \equiv k((p - 3)/2 - \omega + \eta/2) \pmod{q}, \\
\mu & \equiv m((p - 3)/2 - \omega - eq) \pmod{r}.
\end{align*}
\]

We shall now show that if $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions, while in the other three cases (3) has no solutions.

If $\epsilon = \eta = 0$, (4) gives us $\omega \equiv \nu \pmod{p}$ and since both $\omega$ and $\nu$ are less than $p$, $\omega - \nu$. Equations (5) and (6) become in this case
\[
\begin{align*}
\lambda & \equiv k((p - 3)/2 - \nu) \pmod{q}, \\
\mu & \equiv m((p - 3)/2 - \nu) \pmod{r}.
\end{align*}
\]

Since $\nu \leq (p-3)/2$, and $k(p-3)/2$ and $\lambda$ are $<q$, while $m(p-3)/2$ and $\mu$ are $<r$, these congruences are actually equalities, and we have determined for each of the $(p-1)/2$ values of $\nu$, corresponding values of $\lambda$ and $\mu$, which are such that $\lambda \leq k(p-3)/2$, so that
\[
\lambda pr \leq kpr(p - 3)/2 < qr(p - 3)/2 < h,
\]
and $\mu \leq m(p-3)/2$, so that
\[
\mu pq \leq mpq(p - 3)/2 \leq (2r + 1)(p - 3)/2 < h,
\]
so that all the variables are determined within the ranges (2), and hence in the case $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions.

For $\epsilon = 1$, $\eta = 0$, (4) gives us $\omega \equiv v - 2 \pmod{p}$. Hence either $\omega = v - 2$, or $\omega = p - 1$, or $p - 2$. In the last two cases we can use (5) to get
\[
\lambda \equiv k((p - 3)/2 - \omega) \equiv - k(p \pm 1)/2 \pmod{q}.
\]
That is,
\[
\lambda = q - k(p \pm 1)/2 \geq q - k(p - 1)/2,
\]
so that
\[ \lambda pr \geq pqr - kpr(p - 1)/2 > pqr - qr(p - 1)/2 \]
\[ = qr(p + 1)/2 > h. \]
Hence for \( \omega = p - 1 \) or \( p - 2 \), (2) is violated for \( \lambda pr \), and there are no solutions. If \( \omega = v - 2 \leq (p - 7)/2 \), we use (6) and obtain
\[ \mu \equiv m((p - 3)/2 - \omega - q) \quad (\text{mod } r), \]
or
\[ \mu = r + m((p - 3)/2 - \omega - q) \geq r + m(2 - q). \]
Hence
\[ \lambda pq \geq pqr + (2r + 1)(2 - q) \]
\[ = (qr + 1)(p - 2) + (4r - p - q + 4) \]
\[ > (qr + 1)(p - 2) > h, \]
so that (2) is again violated and there are no solutions of (3) for \( \epsilon = 1, \eta = 0 \).

In the next case \( \epsilon = 0, \eta = 1 \), we get from (4) \( \omega = v + (p + 1)/2 \), and putting this value for \( \omega \) in (6), we have
\[ \mu \equiv m((p - 3)/2 - v - (p + 1)/2) \quad (\text{mod } r). \]
Hence \( \mu = r - m(v + 2) \geq r - m(p + 1)/2 \), so that
\[ \lambda pq \geq pqr - (2r + 1)(p + 1)/2 > pqr - (2r + 1)(q - 1)/2 \]
\[ = (qr + 1)(p - 1) + (2r - q - 2p + 3)/2 \]
\[ > (qr + 1)(p - 1) > h. \]
Thus this case does not yield any further solutions. We have now shown that (3) has at least \((p - 1)/2\) solutions, since the remaining case \( \epsilon = \eta = 1 \) would contribute positively, if at all. In fact, it can be shown by a similar reasoning that this case does not contribute any solutions, so that the coefficient of \( x^h \) is precisely \((p - 1)/2\). However, in any case, the coefficient of \( x^h \) increases with \( p \), so that we have proved the following theorem.

**Theorem.** As \( n \) runs over all products of three distinct primes, the cyclotomic polynomials \( Q_n(x) \) contain arbitrarily large coefficients.

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