ON THE MAGNITUDE OF THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

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Until very recently all the results of the investigations into the magnitude of the coefficients of the cyclotomic polynomial

$$Q_n(x) = \prod_{\delta | n} (1 - x^{n/\delta})^{\mu(\delta)} \tag{1}$$

tended to show that these coefficients are very small indeed. In fact for $n<105$ all the coefficients are ± 1, and 0, and for $n<385$ they do not exceed 2 in absolute value.

In 1883 Migotti* showed that the coefficients of $Q_n(x)$ are all ± 1 or 0 for n a product of two primes, but noted that the coefficient of x^7 in $Q_{105}(x)$ is -2. In 1895 Bang† proved that no coefficient of $Q_n(x)$ for $n=pqr$, $(p<q<r$, odd primes), exceeds $p-1$. Nothing further was done on the problem until 1931, when I. Schur gave a very ingenious proof of the following theorem.

SCHUR'S THEOREM. *There exist cyclotomic polynomials with coefficients arbitrarily large in absolute value.*

As this proof has not been published, it is given below.‡

PROOF. Let $n=p_1p_2 \cdots p_t$, where t is odd and $p_1<p_2< \cdots <p_t$ are odd primes such that§ $p_1+p_2>p_t$. To prove the theorem it is sufficient to show that the coefficient of x^{p_t} in $Q_n(x)$ is $1-t$. This can be done by taking $Q_n(x)$ modulo x^{p_t+1}. We then get

$$\begin{aligned} Q_n(x) &\equiv \prod_{i=1}^{t} (1 - x^{p_i})/(1 - x) \\ &\equiv (1 + x + \cdots + x^{p_t - 1})(1 - x^{p_1})(1 - x^{p_2}) \cdots (1 - x^{p_{t-1}}) \\ &\equiv (1 + x + \cdots + x^{p_t - 1})(1 - x^{p_1} - x^{p_2} - \cdots - x^{p_{t-1}}) \\ &\hspace{8cm} (\text{mod } x^{p_t+1}). \end{aligned}$$

* Sitzungsberichte, Akademie der Wissenschaften, Wien. (math), (2), vol. 87 (1883), pp. 7–14.

† Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.

‡ This proof is essentially the one given by Schur in a letter to Landau.

§ Such a set of primes exists for every t.

Collecting the coefficient of x^{pt} in this last expression we see that it is precisely $-(t-1)$, so that as t increases we can exhibit arbitrarily large negative coefficients of the cyclotomic polynomials, which proves the theorem.

The question now remains as to the boundedness of the coefficients of $Q_n(x)$ for a fixed t. We have already seen that for $t=1$ and 2 these coefficients are actually bounded. The case $t=3$ was discussed by Bungers* who proved the following theorem.

BUNGERS' THEOREM. *As n runs over all products of three distinct primes, the cyclotomic polynomials $Q_n(x)$ contain arbitrarily large coefficients, provided there exist infinitely many prime pairs.*

His proof depends on choosing three primes, two of which differ by 2, and in exhibiting a coefficient of $Q_{pqr}(x)$ equal to $(p+1)/2$. It is the purpose of this note to modify Bungers' proof so as to eliminate the unproved assumption of the existence of infinitely many prime pairs.

Let $n=pqr$, where $q=kp+2$, and $r=(mpq-1)/2$. For a given p such primes q and r can always be found by Dirichlet's Theorem. We proceed to show that the coefficient of x^h, where $h=(p-3)(qr+1)/2$ is $(p-1)/2$ and hence can be made arbitrarily large with p. From (1) with $n=pqr$, we have

$$
\begin{aligned}
Q_{pqr}(x) &= \frac{(x^{pqr}-1)(x^p-1)(x^q-1)(x^r-1)}{(x-1)(x^{pq}-1)(x^{pr}-1)(x^{qr}-1)} \\
&\equiv (1+x+\cdots+x^{p-1})(1-x^q-x^r+x^{q+r}) \\
&\quad \cdot \sum x^{\nu qr+\lambda pr+\mu pq} \qquad (\bmod\ x^{pqr}).
\end{aligned}
$$

Since we are interested in the coefficient of x^h, the summation indices ν, λ, μ, satisfy the following inequalities:

$$
(2) \qquad \nu qr \leqq h, \qquad \lambda pr \leqq h, \qquad \mu pq \leqq h.
$$

We now consider the diophantine equation

$$
(3) \quad \nu qr + \lambda pr + \mu pq + \omega + \epsilon q + \eta r = (p-3)(qr+1)/2 = h,
$$

where $\omega < p$, and $\epsilon=0$ or 1, $\eta=0$ or 1.

The coefficient of x^h is now given by the number of solutions of (3) with $\epsilon=\eta$ minus the number of solutions of (3) with $\epsilon\neq\eta$.

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* Göttingen Dissertation, 1934.

Taking (3) modulo p, q, and r we have, since $qr \equiv -1 \pmod{p}$,

$$\begin{aligned} \nu qr + \omega + \epsilon q + \eta r &\equiv 0 \pmod{p}, \\ \lambda pr + \omega + \eta r &\equiv (p-3)/2 \pmod{q}, \\ \mu pq + \omega + \epsilon q &\equiv (p-3)/2 \pmod{r}. \end{aligned}$$

Multiplying the last two congruences by k and m, respectively, and remembering that $kpr \equiv 1 \pmod{q}$, while $mpq \equiv 1 \pmod{r}$, also that $q \equiv 2 \pmod{p}$, and $r \equiv -1/2 \pmod{pq}$, we get

$$\omega \equiv \nu - 2\epsilon + \eta/2 \pmod{p}, \tag{4}$$

$$\lambda \equiv k((p-3)/2 - \omega + \eta/2) \pmod{q}, \tag{5}$$

$$\mu \equiv m((p-3)/2 - \omega - \epsilon q) \pmod{r}. \tag{6}$$

We shall now show that if $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions, while in the other three cases (3) has no solutions.

If $\epsilon = \eta = 0$, (4) gives us $\omega \equiv \nu \pmod{p}$ and since both ω and ν are less than p, $\omega = \nu$. Equations (5) and (6) become in this case

$$\begin{aligned} \lambda &\equiv k((p-3)/2 - \nu) \pmod{q}, \\ \mu &\equiv m((p-3)/2 - \nu) \pmod{r}. \end{aligned}$$

Since $\nu \leqq (p-3)/2$, and $k(p-3)/2$ and λ are $< q$, while $m(p-3)/2$ and μ are $< r$, these congruences are actually equalities, and we have determined for each of the $(p-1)/2$ values of ν, corresponding values of λ and μ, which are such that $\lambda \leqq k(p-3)/2$, so that

$$\lambda pr \leqq kpr(p-3)/2 < qr(p-3)/2 < h,$$

and $\mu \leqq m(p-3)/2$, so that

$$\mu pq \leqq mpq(p-3)/2 \leqq (2r+1)(p-3)/2 < h,$$

so that all the variables are determined within the ranges (2), and hence in the case $\epsilon = \eta = 0$, (3) has $(p-1)/2$ solutions.

For $\epsilon = 1$, $\eta = 0$, (4) gives us $\omega \equiv \nu - 2 \pmod{p}$. Hence either $\omega = \nu - 2$, or $\omega = p - 1$, or $p - 2$. In the last two cases we can use (5) to get

$$\lambda \equiv k((p-3)/2 - \omega) \equiv -k(p \pm 1)/2 \pmod{q}.$$

That is,

$$\lambda = q - k(p \pm 1)/2 \geqq q - k(p-1)/2,$$

so that

$$\begin{aligned}\lambda pr &\geqq pqr - kpr(p-1)/2 > pqr - qr(p-1)/2 \\ &= qr(p+1)/2 > h.\end{aligned}$$

Hence for $\omega = p-1$ or $p-2$, (2) is violated for λpr, and there are no solutions. If $\omega = \nu - 2 \leqq (p-7)/2$, we use (6) and obtain

$$\mu \equiv m((p-3)/2 - \omega - q) \qquad (\bmod r),$$

or

$$\mu = r + m((p-3)/2 - \omega - q) \geqq r + m(2-q).$$

Hence

$$\begin{aligned}\mu pq &\geqq pqr + (2r+1)(2-q) \\ &= (qr+1)(p-2) + (4r - p - q + 4) \\ &> (qr+1)(p-2) > h,\end{aligned}$$

so that (2) is again violated and there are no solutions of (3) for $\epsilon = 1$, $\eta = 0$.

In the next case $\epsilon = 0$, $\eta = 1$, we get from (4) $\omega = \nu + (p+1)/2$, and putting this value for ω in (6), we have

$$\mu \equiv m((p-3)/2 - \nu - (p+1)/2) \qquad (\bmod r).$$

Hence $\mu = r - m(\nu+2) \geqq r - m(p+1)/2$, so that

$$\begin{aligned}\mu pq &\geqq pqr - (2r+1)(p+1)/2 > pqr - (2r+1)(q-1)/2 \\ &= (qr+1)(p-1) + (2r - q - 2p + 3)/2 \\ &> (qr+1)(p-1) > h.\end{aligned}$$

Thus this case does not yield any further solutions. We have now shown that (3) has at least $(p-1)/2$ solutions, since the remaining case $\epsilon = \eta = 1$ would contribute positively, if at all. In fact, it can be shown by a similar reasoning that this case does not contribute any solutions, so that the coefficient of x^h is precisely $(p-1)/2$. However, in any case, the coefficient of x^h increases with p, so that we have proved the following theorem.

THEOREM. *As n runs over all products of three distinct primes, the cyclotomic polynomials $Q_n(x)$ contain arbitrarily large coefficients.*

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