ON POINCARÉ'S RECURRENCE THEOREM

BY CORNELIS VISSEER

1. Introduction. Let $S$ be a space in which is defined a measure $\mu$ such that $\mu(S) = 1$. Suppose we are given a one parameter group of one to one transformations $T_t$, $(-\infty < t < \infty)$, of $S$ into itself, with the properties:

(1) $T_s T_t = T_{t+s}$.

(2) For any measurable set $E$ and any $t$ the set $T_tE$ is measurable and $\mu(T_tE) = \mu(E)$.

The following extension of Poincaré’s recurrence theorem was proved by Khintchine.*

For any measurable $E$ and any $\lambda < 1$,

$$\mu(E \cdot T_tE) \geq \lambda(\mu(E))^2$$

for a set of values $t$ that is relatively dense on the $t$ axis.

In this paper we give an elementary proof of this statement.

2. An Auxiliary Theorem. We prove the following theorem from which the recurrence theorem is an immediate consequence and which is also interesting in itself.

Let $S$ be a space with a measure $\mu$ such that $\mu(S) = 1$ and let $E_1, E_2, \cdots$ be an infinite sequence of measurable sets in $S$, all having a measure not less than $m$. Then for any $\lambda < 1$ there exist in the sequence two sets $E_i$ and $E_k$ such that

$$\mu(E_i E_k) \geq \lambda m^2.$$ 

Let us suppose that $\mu(E_i E_k) < \rho$ for any $i$ and $k$. If we put

$$F_1 = E_1, \quad F_2 = E_2 - E_2 F_1, \quad F_3 = E_3 - E_3 F_2 - E_2 F_1, \quad \cdots, \quad F_n = E_n - E_n F_{n-1} - \cdots - E_2 F_1,$$

no two of the sets $F$ have common points and $F_i$ is part of $E_i$.

Therefore

$$\mu(F_1) \geq m, \quad \mu(F_2) \geq m - \rho, \quad \mu(F_3) \geq m - 2\rho, \quad \cdots, \quad \mu(F_n) \geq m - (n - 1)\rho,$$

and thus for \( n = 1, 2, \ldots \),

\[
\mu(F_1 + \cdots + F_n) = \mu F_1 + \cdots + \mu F_n \geq nm - \frac{1}{2} n(n - 1) \rho.
\]

It follows that

\[
1 = \mu(S) \geq \mu(F_1 + \cdots + F_n) \geq nm - \frac{1}{2} n(n - 1) \rho.
\]

We now choose \( n \) such that

\[
\frac{m}{\rho} < n \leq \frac{m}{\rho} + 1.
\]

Then we obtain

\[
1 \geq \frac{m}{\rho} - \frac{1}{2} \left( \frac{m}{\rho} + 1 \right) \frac{m}{\rho}
\]

\[
= \frac{m^2}{2 \rho} - \frac{m}{2},
\]

or

\[
\rho \geq \frac{m^2}{2 + m}.
\]

Hence, if we exclude the trivial case \( m = 1 \),

\[
\rho > \frac{1}{3} m^2.
\]

From this it follows that there must be two sets \( E_i \) and \( E_k \) with

\[
\mu(E_i E_k) \geq \frac{1}{3} m^2.
\]

We shall now prove that the factor \( 1/3 \) may be replaced by an arbitrary \( \lambda < 1 \). We consider the product space \( S^n \), formed by the systems \((x_1, \ldots, x_n)\) of \( n \) points in \( S \), and in this product space the sequence of sets \( E_1^n, E_2^n, \ldots \). In \( S^n \) we can define a measure \( \mu \) such that the product of \( n \) measurable sets of \( S \) is measurable and has a measure that equals the product of the measures of its components. In applying the result we just ob-
tained to $S^n$ and the sequence $E_1^n$, $E_2^n$, $\cdots$, we find two sets $E_i^n$ and $E_k^n$ such that

$$\mu(E_i^n E_k^n) \geq \frac{1}{3} (m^n)^2.$$ 

Now

$$\mu(E_i^n E_k^n) = \mu((E_i E_k)^n) = (\mu(E_i E_k))^n,$$ 

and consequently

$$(\mu(E_i E_k))^n \geq \frac{1}{3} (m^n)^2,$$ 

or

$$\mu(E_i E_k) \geq \left( \frac{1}{3} \right)^{1/n} m^n.$$

Given $\lambda < 1$, we can always define $n$ such that $(1/3)^{1/n} \geq \lambda$ and then select the pair $E_i$, $E_k$. This proves the theorem.

3. **Proof of the Recurrence Theorem.** Assume the contrary: There is a measurable set $E$ and a number $\lambda < 1$ such that

$$(*) \quad \mu(E \cdot T_i E) < \lambda(\mu(E))^2$$

on arbitrarily large $t$-intervals. Let $I_1$ be a closed interval on which $(*)$ holds; denote by $2l_1$ its length and by $l_1$ its center. There is an interval $I_2$ on which $(*)$ holds and which has a length $> 2(l_1 + |l_2|)$. Denote by $l_3$ the center of $I_2$ and by $I_3$ an interval on which $(*)$ holds and which has a length $> 2(l_1 + |l_2| + |l_3|)$, and so forth. Then the numbers $l_k - l_i$, $(i < k)$, belong to the intervals $I_{k-i}$; hence for any $i$ and $k$, $(i < k)$,

$$\mu(E \cdot T_{i_k-i}E) < \lambda(\mu(E))^2,$$

and consequently

$$\mu(T_i E \cdot T_{i_k} E) < \lambda(\mu(E))^2$$

in contradiction to the theorem of §2, applied to the sequence $T_{i_k}E$, $T_{i_{k-1}E}$, $\cdots$. This proves the recurrence theorem.

It will be seen that it is not essential that $t$ in $T_t$ is a continuous parameter. The same method gives the same result in the case that $t$ only runs through the values $0, \pm 1, \pm 2, \cdots$. 

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4. Remark. Let the sequence $E_1, E_2, \cdots$ be as in §2. Then we can even assert that for every $\lambda < 1$ there exists an infinite subsequence $E_{i_1}, E_{i_2}, \cdots$ such that for every $p$ and $q$

$$\mu(E_{i_p} E_{i_q}) \geq \lambda m^2.$$ 

We show first that there exists an infinite subsequence $E_{k_1}, E_{k_2}, \cdots$ such that $\mu(E_{k_1}, E_{k_p}) \geq \lambda m^2$ for every $p$. Suppose that no such subsequence exists; then to every $n = 1, 2, \cdots$ belongs a $p_n$ such that

$$\mu(E_n E_m) < \lambda m^2 \text{ for } m \geq n + p_n.$$ 

Writing $n_1 = 1$, $n_2 = n_1 + p_{n_1}$, $n_2 = n_2 + p_{n_2}$, \cdots, we have then for every $i$ and $k$,

$$\mu(E_{n_i} E_{n_k}) < \lambda m^2,$$

which contradicts the theorem of §2. The proof is now easily completed by applying the diagonal principle.

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ON THE ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION*

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1. Introduction. The primary object of this note is to give a simple solution of a problem already discussed by many authors including the present one.† It is the problem of determining the regions within which lie the zeros of the derivative of a rational function when the zeros and poles of the function lie in prescribed circular regions.

Theorem 1.‡ For $j = 0, 1, \cdots$, $p$ let $r_j$ and $\sigma_j$ be real constants

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* Presented to the Society, September 4, 1934.
† For an expository account and list of references see M. Marden, American Mathematical Monthly, vol. 42 (1935), pp. 277–286, hereafter referred to as Marden I.
‡ See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 81–109, hereafter referred to as Marden II.