4. Remark. Let the sequence $E_1, E_2, \cdots$ be as in §2. Then we can even assert that for every $\lambda < 1$ there exists an infinite subsequence $E_{i_1}, E_{i_2}, \cdots$ such that for every $p$ and $q$
\[
\mu(E_{i_p} E_{i_q}) \geq \lambda m^2.
\]

We show first that there exists an infinite subsequence $E_{k_1}, E_{k_2}, \cdots$ such that $\mu(E_{k_1}, E_{k_p}) \geq \lambda m^2$ for every $p$. Suppose that no such subsequence exists; then to every $n=1, 2, \cdots$ belongs a $p_n$ such that

\[
\mu(E_n E_{m}) < \lambda m^2 \text{ for } m \geq n + p_n.
\]

Writing $n_1 = 1$, $n_2 = n_1 + p_{n_1}$, $n_3 = n_2 + p_{n_2}$, $\cdots$, we have then for every $i$ and $k$,

\[
\mu(E_{n_i} E_{n_k}) < \lambda m^2,
\]

which contradicts the theorem of §2. The proof is now easily completed by applying the diagonal principle.

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ON THE ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY MORRIS MARDEN

1. Introduction. The primary object of this note is to give a simple solution of a problem already discussed by many authors including the present one.† It is the problem of determining the regions within which lie the zeros of the derivative of a rational function when the zeros and poles of the function lie in prescribed circular regions.

Theorem 1.‡ For $j = 0, 1, \cdots, p$ let $r_j$ and $\sigma_j$ be real constants

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† For an expository account and list of references see M. Marden, American Mathematical Monthly, vol. 42 (1935), pp. 277–286, hereafter referred to as Marden I.
‡ See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 81–109, hereafter referred to as Marden II.
with \( \sigma_j^2 = 1 \); let \( Z_j \) denote the circular region defined by the inequality
\[
\sigma_j Z_j(z) = \sigma_j(\abs{z - \alpha_j}^2 - r_j^2) \leq 0,
\]
and let \( z_i \) be an arbitrary point of the region \( Z_j \). Then every zero of the derivative of the function\(*
\[
f(z) = \prod_{j=0}^{p} (z - z_j)^{m_j}
\]
satisfies at least one of the \( p + 2 \) inequalities
\[
\begin{align*}
& (1) \quad \sigma_j Z_j(z) \leq 0, \quad (j = 0, 1, \ldots, p), \\
& (2) \quad \frac{Z(z)}{\prod_{j=0}^{p} Z_j(z)} = \left( \sum_{j=0}^{p} \frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)} \right)^2 - \left( \sum_{j=0}^{p} \frac{m_j \sigma_j r_j}{Z_j(z)} \right)^2 \leq 0.
\end{align*}
\]

Theorem 1 holds even when the \( m_j \) are complex numbers. When, however, they are all real, the use of the identities
\[
(\alpha_j - z)(\bar{\alpha}_k - \bar{z}) + (\bar{\alpha}_j - \bar{z})(\alpha_k - z) = \abs{\alpha_j - z}^2 + \abs{\alpha_k - z}^2 - \abs{\alpha_j - \alpha_k}^2,
\]
and
\[
\abs{\alpha_j - z}^2 = Z_j(z) + r_j^2,
\]
enables one, after expanding (2), to write
\[
\begin{align*}
& (3) \quad \frac{Z(z)}{\prod_{j=0}^{p} Z_j(z)} = \sum_{j=0}^{p} \frac{nm_j}{Z_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{Z_j(z)Z_k(z)},
\end{align*}
\]
where \( n = \sum m_j \) and
\[
\tau_{jk} = \abs{\alpha_j - \alpha_k}^2 - (\abs{m_j} \abs{m_k}^{-1} \sigma_j r_j) - (\abs{m_k} \abs{m_k}^{-1} \sigma_k r_k)^2.
\]
The latter is the square of the common external or internal tangent of the circles \( Z_j(z) = 0 \) and \( Z_k(z) = 0 \) according as the product \( \abs{m_j m_k} (m_j m_k)^{-1} \sigma_j \sigma_k \) is positive or negative. The equation \( Z(z) = 0 \) represents for \( n \neq 0 \) a \( p \)-circular \( 2p \)-ic and for \( n = 0 \) in general a \( (p-1) \)-circular \( 2(p-1) \)-ic curve. The properties of these curves are studied in Marden II, p. 92.

\* Where no limits are indicated, a product or summation is to be taken from \( j = 0 \) to \( j = p \) and from \( k = j+1 \) to \( k = p \).
2. Three Lemmas.* (I). If the points \( t_0, t_1, \ldots, t_p \) varying independently of one another describe the closed interiors of the circles \( T_0, T_1, \ldots, T_p \), respectively, the center and radius of \( T_j \) being \( \gamma_j \) and \( \rho_j \), then the point \( w = \sum_{j=0}^{p} m_j t_j \) describes the closed interior of a circle \( W \) with center at \( \gamma = \sum_{j=0}^{p} m_j \gamma_j \) and radius of \( \rho = \sum_{j=0}^{p} m_j |\rho_j| \).

For
\[
| w - \gamma | = \left| \sum_{j=0}^{p} m_j (t_j - \gamma_j) \right| \leq \rho;
\]
conversely, if \( k \) and \( \theta \) are arbitrary, \( 0 \leq k \leq 1 \), and if \( m_j (t_j - \gamma_j) = k |m_j| \rho_j e^{i\theta} \), then \( w - \gamma = \sum k |m_j| \rho_j e^{i\theta} = kpe^{i\theta} \).

(II). If the points \( t_j, (j \geq 1) \), vary as in Lemma (I), but the point \( t_0 \) describes the closed exterior of its circle \( T_0 \), the locus of the point \( w \) is the closed exterior of a circle with center at \( \gamma = \sum_{j=0}^{p} m_j \gamma_j \) and radius \( \rho = 2 |m_0| \rho_0 - \sum_{j=0}^{p} m_j |\rho_j| \) provided \( \rho > 0 \), and is the entire plane if \( \rho \leq 0 \).

For, when \( \rho > 0 \),
\[
| w - \gamma | \geq | m_0 (t_0 - \gamma_0) | - \left| \sum_{j=1}^{p} m_j (t_j - \gamma_j) \right| \geq \rho;
\]
conversely, if \( k \) and \( \theta \) are arbitrary with \( k \geq 1 \), and if
\[
m_0 (t_0 - \gamma_0) = \left| m_0 \right| \rho_0 + (k - 1) \rho e^{i\theta},
\]
and
\[
m_j (t_j - \gamma_j) = - |m_j| \rho_j e^{i\theta}, \quad (j \geq 1),
\]
then \( w - \gamma = kpe^{i\theta} \).

If \( \rho_0 \) decreases while \( \rho_j, (j \geq 1) \), remain constant, \( \rho \) will approach zero and the locus of \( w \) will become the entire plane. The locus is, therefore, the entire plane for \( \rho \leq 0 \).

(III). If the points \( t_j, (j > k \geq 1) \), vary as in Lemma (I), but the points \( t_j, (j \leq k) \), describe the exteriors of their circles \( T_j \), the locus \( W \) of \( w \) is the entire plane.

For, if each \( t_j, (1 \leq j \leq k) \), were to vary merely interior to a

circle $T'_j$ drawn exterior to but not enclosing the circle $T_j$ while the remaining $t_i$ vary as indicated in Lemma (III), and if the radius $p'_j$ of $T'_j$ were chosen so that

$$|m_0| \rho_0 - \sum_{j=1}^k |m_j| \rho'_j - \sum_{j=k+1}^p |m_j| \rho_j = 0,$$

then by Lemma (II) the locus of $w$ would be the entire plane.

3. Proof of Theorem 1. Let $z$ be any fixed point exterior to all the regions $Z_j$; that is, let $z$ be such that $\sigma_j Z_j(z) > 0$, all $j$. Since $\sigma_j Z_j(z_j) \leq 0$, point $t_j = (z_j - \alpha_j)^{-1}$ lies in or on the circle $T_j$ with center $\gamma_j = (\alpha_j - \bar{z})/Z_j(z)$ and radius $\rho_j = (\rho_j \gamma_j)/Z_j(z)$. According to Lemma (I), the locus $W_z$ of the point $w = \sum m_j t_j$ will be defined by the inequality:

$$|w - \sum_{j=0}^p m_j (\alpha_j - \bar{z})/Z_j(z)|^2 - \left( \sum_{j=0}^p |m_j| \sigma_j \gamma_j \right)^2 \leq 0. \tag{4}$$

Now, in order to be a zero of $f'(z)$, point $z$ must be a root of the equation

$$-f'(z)/f(z) = \sum_{j=0}^p m_j (z_j - \bar{z})/Z_j(z) = 0; \tag{5}$$

that is, point $w = 0$ must satisfy inequality (4). Hence, any zero of $f'(z)$, not satisfying any of the inequalities (1), must satisfy (2); that is to say, $\sigma Z(z) \leq 0$, where $\sigma = \prod \sigma_j$.

4. A Locus Problem. What then is the locus $Z$ of the zeros of $f'(z)/f(z)$ when the points $z_j$ vary independently within their circular regions $Z_j$?

Theorem 1 reveals that $\sigma Z(z) \leq 0$ for any point $z$ of locus $Z$ exterior to all the regions $Z_j$. Conversely, if exterior to all the regions $Z_j$, any point $z$ for which $\sigma Z(z) \leq 0$ belongs to the locus $Z$. With the aid of Lemma (II), it can be shown that a point $z$ interior to just one region $Z_j$ belongs to locus $Z$ if and only if either $\sigma Z(z) \leq 0$ or $\sigma S(z) \leq 0$, where

$$\frac{S(z)}{\prod Z_j(z)} = \sum_{j=0}^p \frac{|m_j| r_j \sigma_j}{Z_j(z)}.$$

(The curve $S(z) = 0$, in general a $p$-circular $2p$-ic, consists only of points on the boundaries of two or more regions $Z_j$ or interior
to at least one region \(Z_i\), the points \(z\) interior to just one region \(Z_i\) satisfying inequality \(\sigma Z(z) \leq 0\). Likewise, with the aid of Lemma (III), it can be shown that every point common to two or more regions \(Z_i\) belongs to locus \(Z\).

If a given point \(z\) is to be on the boundary of locus \(Z\), point \(w=0\) must cease to be a point of \(W\), whenever the regions \(Z_i\) and hence \(W_z\) are diminished, no matter how slightly. That is to say, the point \(w=0\) must be on the boundary of the locus \(W_z\). This implies that \(Z(z) = 0\) if \(z\), a boundary point of locus \(Z\), is exterior to all the regions \(Z_i\) or interior to just one region \(Z_i\) with \(\sigma Z(z) \leq 0\). It also implies that no boundary point \(z\) of locus \(Z\) may be either interior to just one region \(Z_i\) with \(\sigma S(z) \leq 0\), or interior to two or more regions \(Z_i\); for, in those cases, the locus \(W_z\) is the entire plane.

In short, the locus \(Z\) is a set of regions bounded by the ovals of the curve \(Z(z) = 0\), each region, according to Marden II, being simply-connected.

5. Applications. When \(p = 2\) and \(n = m_0 + m_1 + m_2 = 0\), equation (5) may be written as the cross-ratio

\[
(z_0 z_1 z_2 z) \equiv \frac{(z_0 - z_2)(z_1 - z)}{(z_0 - z)(z_1 - z_2)} = -\frac{m_1}{m_0}.
\]

Here

\[
Z(z) = -m_0 m_1 r_0 Z_2(z) - m_1 m_2 r_1 Z_0(z) - m_2 m_0 r_2 Z_1(z) = 0,
\]

\[
S(z) = |m_0| r_0 \sigma_0 Z_1(z) Z_2(z) + |m_1| r_1 \sigma_1 Z_2(z) Z_0(z) + |m_2| r_2 \sigma_2 Z_0(z) Z_1(z) = 0,
\]

represent in general a circle and bicircular quartic, respectively. If \(\lambda\) denotes the coefficient of the term \((x^2 + y^2)^2\) in the expression \(Z(z)\), the region \(\sigma Z(z) \leq 0\) is the interior or exterior of the circle \(Z(z) = 0\) according as \(\sigma \lambda > 0\) or \(\sigma \lambda < 0\). Hence, if all the points for which \(\sigma S(z) \leq 0\) lie in the region \(\sigma Z(z) \leq 0\), the locus \(Z\) will be the interior or exterior of circle \(Z(z) = 0\) according as \(\sigma \lambda > 0\) or \(\sigma \lambda < 0\). If, however, not all the points for which \(\sigma S(z) \leq 0\) lie in the region \(\sigma Z(z) \leq 0\), the locus will be the entire plane. This discussion verifies the following theorem due to Walsh.*

If the points \( z_0, z_1, \) and \( z_2 \) varying independently of one another describe given circular regions \( Z_0, Z_1, \) and \( Z_2, \) then the point \( z \) defined by the constant cross-ratio \( (z_0 z_1 z_2 z) = c \) also describes a circular region \( Z. \)

On allowing a number of the regions \( Z_i \) to coincide, we deduce from Theorem 1 the following corollary.

**Corollary 1.** If all the zeros of a polynomial \( f_j(z) \) of degree \( n_j \) lie in the circular region \( Z_j, \) then every zero of the derivative of the product

\[
\prod_{j=0}^{p} [f_j(z)]^{q_j}
\]

will satisfy at least one of the \( p + 2 \) inequalities (1) and (2) with \( m_j = n_j q_j. \)

In particular, upon setting \( f_j(z) = f(z) - \gamma_j, \) we obtain from Corollary 1 a generalization of a theorem stated by Jentsch and proved by Fekete.

**Corollary 2.** If all the points at which a given polynomial \( f(z) \) takes on the value \( \gamma_j \) lie in the circular region \( Z_j, \) then every root \( z \) of the equation

\[
\sum_{j=0}^{p} \frac{m_j}{f(z) - \gamma_j} = 0
\]

satisfies at least one of the \( p + 2 \) inequalities (1) and (2).

6. **Generalizations.** By requiring \( w = \lambda \) instead of \( w = 0 \) to satisfy inequality (4), we are led to the following result.

**Theorem 2.** Under the hypotheses of Theorem 1 or of Corollary 1, every zero of the function \( f'(z) + \lambda f(z) \) satisfies at least one of the \( p + 2 \) inequalities

\[
\sigma_j Z_j(z) \leq 0, \quad (j = 0, 1, \ldots, p),
\]

\[
\lambda - \sum_{j=0}^{p} \frac{m_j (\alpha_j - \bar{z})}{Z_j(z)} \right\}^2 - \left( \sum_{j=0}^{p} \frac{|m_j| \sigma_j}{Z_j(z)} \right)^2 \leq 0.
\]

* This corollary contains as special cases a number of important theorems due to Gauss-Lucas, Laguerre, Bôcher, and Walsh. See Marden I.

If all the $m_i$ are real, the left-hand side of (8) may be rewritten, with the aid of the identities given in §1, as

$$
\sum_{j=0}^{p} \frac{\lambda |m_i \Gamma_j(z)|}{nZ_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{Z_j(z)Z_k(z)},
$$

where

$$
\Gamma_j(z) \equiv |z - (\alpha - n\lambda^{-1})|^2 - r_j^2.
$$

The equation $\Gamma_j(z) = 0$ represents the circle obtained by translating the circle $Z_i(z) = 0$ in the direction and magnitude of the vector $n/\lambda$. Set equal to zero, expression (7) represents a $(p+1)$-circular $2(p+1)$-ic curve with singular foci at the roots of the equation

$$
\lambda + \sum_{j=0}^{p} \frac{m_i}{z - \alpha_j} = 0.
$$

In particular, assuming the hypotheses of Corollary 1, and setting $\sigma_0 - 1 = p = p_0 - 1 = 0$, we find this curve to reduce to the circle $\Gamma_0(z) = 0$. In other words, if all zeros of a polynomial $f(z)$ of degree $n$ lie in a given circle, any zero of the linear combination $f'(z) + \lambda f(z)$ will lie either in the given circle or in the one obtained by translating the given circle in the direction and magnitude of the vector $n\lambda^{-1}$. *

Finally, by requiring $w = g(z)$, an arbitrary function of $z$, instead of $w = 0$, to satisfy inequality (4), we obtain a theorem similar to Theorem 2 for the zeros of the function $f'(z) + g(z)f(z)$. For example, if $g(z) = \bar{z}$, and if all the zeros of a polynomial $f(z)$ of degree $n$ lie in the circle $|z| \leq r \leq 2n^{1/2}$, all zeros of the function $\bar{z}f(z) + f'(z)$ lie in the same circle.

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