ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

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1. Introduction. A set \( \phi_0(x) = 1, \phi_1(x), \phi_2(x) \) of polynomials of degrees 0, 1, 2, \( \cdots \), is called a set of orthogonal polynomials if they satisfy

\[
\int_a^b \phi_m(x)\phi_n(x)d\psi(x) = 0, \quad \int_a^b d\psi(x) > 0, \quad (m \neq n),
\]

where \( \psi(x) \) is a non-decreasing function of bounded variation. There is no restriction in assuming the highest coefficient 1.

It has been shown by W. Hahn\( ^\dagger \) that if the derivatives also form a set of orthogonal polynomials, then the original set were Jacobi, Hermite, or Laguerre polynomials. His method consisted in showing that the polynomials satisfy a differential equation of the type

\[
(a + bx + cx^2)\phi_n'' + (d + ex)\phi_n' + \lambda_n\phi_n = 0.
\]

From this it followed that the set were Jacobi, Hermite, or Laguerre polynomials.

Here we propose to give a new proof of this result, our point of view being to answer the question: What conditions on the weight function result from assuming that both \( \{\phi_n(x)\} \) and \( \{\phi'_n(x)\} \) are sets of orthogonal polynomials? However, we shall assume that \( (a, b) \) is a finite interval and \( d\psi(x) = \rho(x)dx \),\( ^\ddagger \) where the weight function is \( L \)-integrable.

2. A Relation for the Weight Function \( \rho(x) \). Let the set \( \{\phi'_n(x)\} \) be orthogonal in the interval \( (c, d) \), infinite or not, with the weight function \( \rho(x) \), that is,

\[
\int_c^d \rho(x)\phi'_n(x)\phi'_m(x)dx = 0, \quad (m \neq n).
\]

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\( ^\ddagger \) Professor Shohat informs me that he has discussed the general case \( d\psi(x) \).
The polynomials \( \{\phi_n(x)\} \), \( \{\phi'_n(x)\} \) satisfy the recurrence relations,*

\[
\phi_{n+2}(x) = (x - c_{n+2})\phi_{n+1}(x) - \lambda_{n+2}\phi_n(x),
\]

\[
\frac{1}{n + 2} \phi'_{n+2}(x) = \frac{x - c'_{n+2}}{n + 1} \phi'_{n+1}(x) - \lambda'_{n+2}\phi'_n(x),
\]

\[(n \geq 0; c_n, c'_n, \lambda_n, \lambda'_n \text{ constants}).\]

Differentiating both sides of the first relation and eliminating the term containing \( x \), by means of the second relation, we get

\[
\phi_{n+1}(x) = \frac{1}{n + 2} \phi'_{n+2}(x) + c''_{n+2}\phi'_{n+1}(x) + \lambda''_{n+2}\phi'_n(x),
\]

\[(c''_n, \lambda''_n \text{ constants}).\]

Remembering that \( \phi'_n(x) \), with the weight function \( q(x) \), is orthogonal to any polynomial of degree \( \leq n - 2 \), we get

\[
(1) \int q(x)\phi'_{n+1}(x)G_{n-2}(x)dx = 0,
\]

where \( G_n(x) \) is an arbitrary polynomial of degree \( \leq n \).

**Lemma.** Let \( Q(x) \) be non-negative in \( (c, d) \), and such that the numbers

\[
\beta_k = \int_c^d Q(x)x^kdx, \quad (k = 0, 1, \ldots),
\]

exist, and for a certain positive integer \( r \)

\[
(2) \int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \quad (n = r + 1, r + 2, \ldots).
\]

Then almost everywhere

\[
Q(x) = \begin{cases} P_r(x)p(x) & \text{in } (a, b), \\ 0 & \text{elsewhere}, \end{cases}
\]

where \( P_r(x) \) is a polynomial of degree \( \leq r \).

Consider the function

We determine the \( \{u_i\} \) so that

\[
\int_a^b R(x) x^i dx = \int_c^d Q(x) x^i dx, \quad (i = 0, 1, \ldots, r);
\]

that is, the \( u_i \) satisfy the equations

\[
\begin{align*}
\alpha_0 u_0 + \alpha_1 u_1 + \cdots + \alpha_r u_r &= \beta_0, \\
\alpha_1 u_0 + \alpha_2 u_1 + \cdots + \alpha_{r+1} u_r &= \beta_1, \\
&\quad \ldots \ldots \ldots \ldots \ldots \ldots \\
\alpha_r u_0 + \alpha_{r+1} u_1 + \cdots + \alpha_{2r} u_r &= \beta_r,
\end{align*}
\]

where

\[
\alpha_k = \int_a^b x^k p(x) dx.
\]

This is always possible for the determinant of this system is known to be positive.\(^*\) Now\(^\dagger\)

\[
0 = \int_a^b R(x) \phi_{r+i}(x) dx = \int_c^d Q(x) \phi_{r+i}(x) dx, \quad (i = 1, 2, \ldots).
\]

And then

\[
\int_a^b R(x) x^i dx = \int_c^d Q(x) x^i dx, \quad (i = 0, 1, 2, \ldots).
\]

Let

\[
f(x) = \begin{cases} 
Q(x) - R(x) & \text{in } E_1, \text{ the points where } (a, b) \text{ and } (c, d) \text{ overlap,} \\
R(x) & \text{in } E_2, \text{ the remainder of } (a, b), \\
Q(x) & \text{in } E_3, \text{ the remainder of } (c, d).
\end{cases}
\]

Then

\[
\int_{E_1+E_2+E_3} f(x) x^i dx = 0, \quad (i = 0, 1, 2, \ldots).
\]

If \((c, d)\) is finite, \(f(x)\) must be zero almost everywhere, and our

\(^*\) See Shohat, loc. cit., p. 9, formula (19).

\(^\dagger\) As a special case, take in (2), \(n = r + i\); that is, \(\int_c^d Q(x) \phi_{r+i}(x) G_{r-1}(x) dx = 0\), then take \(G_{r-1}(x) = 1\).
lemma follows. Moreover, from the above definition of \( f(x) \), we conclude that \( R(x) \), hence \( p(x) \), \( \equiv 0 \) almost everywhere in \( E_3 \), so that both intervals \((a, b)\) and \((c, d)\) may be reduced to their common part \( E_1 \); in other words, here we may take \((c, d) \equiv (a, b)\). If \((c, d)\) is infinite, let \( A \) = \( \max (|a|, |b|) \), so that \( |x| \leq A \) in \( E_1 + E_2 \), which is identical with \((a, b)\), and let \( E_4 \) be the part of \( E_3 \) for which \( |x| \geq (1+\alpha)A \), where \( \alpha > 0 \) is arbitrary. If \( Q(x) \geq 0 \), we have

\[
(1 + \alpha)A \int_{E_4} Q(x) dx \leq \int_{E_3} Q(x) x^i dx = \left| \int_{E_{1+E_2}} f(x) x^i dx \right|
\]

\[
\leq A \int_{E_{1+E_2}} |f(x)| dx,
\]

\[
(1 + \alpha)^i \leq \frac{\int_{E_{1+E_2}} |f(x)| dx}{\int_{E_4} Q(x) dx},
\]

for all even \( i \), which is impossible unless \( Q(x) \) is zero almost everywhere in \( E_4 \). Hence \((c, d)\) reduces to \( E_1 + E_3 - E_4 \) and in \((c, d), \ |x| < (1+\alpha)A \); that is, in view of the arbitrariness of \( \alpha(>0) \),

\[
| x | \text{ in } (c, d) \leq A = \max (|a|, |b|),
\]

which requires that \((c, d) \equiv (a, b)\). The following important result has been established: \((c, d)\) is finite and coincides with \((a, b)\).

With this lemma, it follows from (1) that

\[
q(x) = (rx^2 + sx + t)p(x),
\]

and we may take \( c = a, d = b \).

3. **Existence of \( q'(x) \).** Consider the function

\[
S(x) = k \int_a^x (x - l) p(x) dx,
\]

where \( k \) and \( l \) are such that \( S(b) = 0, \int_a^b S(x) dx = \int_a^b q(x) dx \). An integration by parts applied to \( \int_a^b S(x) \phi_{n+1}^\prime(x) dx \) gives, since \( S(a) = S(b) = 0 \),

\[
\int_a^b S(x) \phi_{n+1}^\prime(x) dx = \int_a^b \phi_{n+1}(x) k(x - l) p(x) dx = 0, \quad (n \geq 1).
\]
But \( q(x) \) is the weight function for the orthogonal polynomials \( \{ \phi'_n(x) \} \), whence
\[
\int_a^b S(x) \phi'_{n+1}(x) \, dx = \int_a^b q(x) \phi'_{n+1}(x) \, dx = 0, \quad (n \geq 1).
\]
This and the relation \( \int_a^b S(x) \, dx = \int_a^b q(x) \, dx \) gives
\[
\int_a^b S(x) x^n \, dx = \int_a^b q(x) x^n \, dx, \quad (n \geq 0),
\]
and then \( q(x) = S(x) \) almost everywhere. Since \( S(x) \) has a derivative almost everywhere, \( q(x) \) has a derivative almost everywhere and
\[
q'(x) = k(x - l) p(x), \quad q(a) = q(b) = 0.
\]

4. Discussion of \( q(x) \) and \( p(x) \). Dividing (4) by (3), we get
\[
\frac{q'(x)}{q(x)} = \frac{k(x - l)}{r x^2 + s x + t}, \quad q(a) = q(b) = 0.
\]
We proceed to show (i) \( r x^2 + s x + t \) has real zeros, (ii) \( r \neq 0 \). (i) Assume \( r x^2 + s x + t \) has imaginary zeros. Integrating the differential equation (5), we get
\[
\log q(x) = \int \frac{k(x - l)}{r x^2 + s x + t} \, dx + c,
\]
\[
q(x) = K (r x^2 + s x + t)^{\alpha \beta} \arctan (r x + 4) ,
\]
\[
(\alpha, \beta, \gamma, \delta, K \text{ constants}).
\]
This is incompatible with \( q(a) = q(b) = 0 \). (ii) Assume first \( r = s = 0 \). Equation (5) becomes
\[
q'(x) = (2 \alpha x + \beta) q(x),
\]
\[
q(x) = K e^{\alpha x + \beta x}, \quad (\alpha, \beta, K \text{ constants}),
\]
which is not zero at \( a \) and \( b \).

Next, suppose \( r = 0, s \neq 0 \). Equation (5) gives
\[
\frac{q'(x)}{q(x)} = \frac{k(x - l)}{s x + t} = \alpha + \frac{\beta s}{s x + t},
\]
\[
q(x) = K (s x + t)^{\beta} e^{\alpha x}, \quad (\alpha, \beta, K \text{ constants}).
\]
This cannot vanish at both end points, \( x = a \) and \( x = b \).

Having thus proved (i) and (ii), we set
\[
rx^2 + sx + t = r(x - g)(x - h), \quad (r \neq 0, g, h \text{ real}),
\]
and rewrite (5) as follows:
\[
\frac{q'(x)}{q(x)} = \frac{k(x - l)}{rx^2 + sx + t} = \frac{\alpha}{x - g} + \frac{\beta}{x - h},
\]
whence
\[
q(x) = K(x - g)^\alpha(x - h)^\beta, \quad (K, \alpha, \beta, \text{constants}).
\]
The conditions \( q(a) = q(b) = 0 \) demand that \( g = a, h = b \), so that finally (disregarding inconsequential constant factors)
\[
q(x) = -r(x - a)^\alpha(b - x)^\beta.
\]
And then from (3)
\[
\rho(x) = \frac{q(x)}{rx^2 + sx + t} = \frac{r(x - a)^\alpha(b - x)^\beta}{r(x - a)(b - x)},
\]
\[
\rho(x) = (x - a)^{\alpha - 1}(b - x)^{\beta - 1},
\]
and we can see that \( \alpha, \beta \) are both >0.

5. Conclusion. Since this is the weight function for Jacobi polynomials, we have thus established the following theorem.

**Theorem.** If \( \{ \phi_n(x) \} \) is a set of orthogonal polynomials with the weight function \( \rho(x) \) in the finite interval \( (a, b) \), and if we assume that the derivatives \( \{ \phi'_n(x) \} \) also form a set of orthogonal polynomials in a certain interval \( (c, d) \) (infinite or not), with a non-negative weight function \( q(x) \), then \( \{ \phi_n(x) \} \) is a set of Jacobi polynomials.

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