A CHARACTERIZATION OF MANIFOLD BOUNDARIES IN $E^n$ DEPENDENT ONLY ON LOWER DIMENSIONAL CONNECTIVITIES OF THE COMPLEMENT*

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In my recent paper Generalized closed manifolds in n-space† it was shown‡ that a compact point set $B$ in $E_n$, common boundary of (at least) two domains $D$ and $D_1$ which are respectively u.l.i.c.§ for $0 \leq i \leq j$ and $0 \leq i \leq n-j-3$ (where $n-2 > j \geq (n-3)/2$), and such that the Betti numbers $p_i^1(D)$, $p_i^{i+2}(D)$, $\ldots$, $p_i^{n-2}(D)$ are finite, is a g.c.$(n-1)$-m. This constituted a generalization of a former result|| to the effect that when $n = 3$, $D$ and $D_1$ are u.l.i.c., and $p_i(D)$ is finite, $B$ is a closed 2-manifold. In the present note I propose to show, as principal result, that the above conditions on the numbers $p_i^{i+2}(D)$, $\ldots$, $p_i^{n-2}(D)$ are irrelevant, and furthermore that it is immaterial whether we place the restriction as to finiteness on $p_i^{i+1}(D)$ or on $p_i^{n-i-2}(D_1)$. It turns out that the only essential requirements are that the upper limits on the dimensions for which $D$ and $D_1$ are u.l.i.c. must total at least $n-3$, and that one of the domains have a finite Betti number as just stated.

For the sake of brevity we make the following definitions. We shall understand without explicit statement hereafter that the imbedding space is $E_n(n \geq 3)$ (euclidean space of $n$ dimensions).

**DEFINITION.** A metric space will be said to be completely $i$-avoidable¶ at a point $P$ if for every $\epsilon > 0$ there exist $\delta$ and $\eta$, $\epsilon > \delta > \eta > 0$, such that if $\gamma^i$ is a cycle on $F(P, \delta)$, then $\gamma^i \sim 0$ on $S(P, \epsilon) - S(P, \eta)$.

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* Presented to the Society, December 29, 1934.
† Annals of Mathematics, vol. 35 (1934), pp. 876–903; to be referred to hereafter as G.C.M.
‡ Principal Theorem E of G.C.M.
§ u.l.i.c. = uniformly locally $i$-connected; see G.C.M. for definition.
¶ See condition (3), definition $M_i^a$, of G.C.M.
THEOREM 1. Let $M$ be the boundary of a u.l.i-c. domain $D$, $(0 \leq i \leq n-j-2)$, and $P$ a point of $M$ at which $M$ is completely $(n-j-2)$-avoidable. Then there exists for every $\epsilon > 0$ an $\eta > 0$ such that if $\gamma^i \subset S(P, \eta)$ links $M$, then $\gamma^i$ is linked with a cycle $\Gamma^{n-i-1}$ of $D \cdot S(P, \epsilon)$, and with a cycle $\Gamma^{n-i-1}$ of $M \cdot S(P, \epsilon)$.*

PROOF. Let $\epsilon'$ be an arbitrary positive number $< \epsilon$, and let $\delta$ and $\eta$ be such that a $\gamma^{n-i-2}$ of $M \cdot F(P, \delta)$ is homologous to zero on $M \cdot [S(P, \epsilon') - S(P, \eta)]$. Suppose $\gamma^i \subset S(P, \eta)$ links $M$. Let $H = M \cdot S(P, \delta)$ and $K = M \cdot S(P, \epsilon')$. Then $\gamma^i$ links $K$. For suppose not. Then there exists a chain $C_i^{i+1} \rightarrow \gamma^i$ in $E_n - K$, and hence in $E_n - H$. A chain $C_1^{i+1} \rightarrow \gamma^i$ in $S(P, \eta)$ lies also in $E_n - (M - H)$. Then the cycle $C_1^{i+1} - C_2^{i+1}$ must link $H \cdot M - H$, else by the Alexander Addition Theorem $\gamma^i$ does not link $H + M - H = M$. But then $C_1^{i+1} - C_2^{i+1}$ is linked with an $(n-j-2)$-cycle of $M \cdot F(P, \delta)$, since $H \cdot M - H \subset M \cdot F(P, \delta)$. But such a cycle bounds on $M \cdot [S(P, \epsilon') - S(P, \eta)] \subset E_n - C_1^{i+1} - C_2^{i+1}$,† and we have a contradiction. Thus $\gamma^i$ links $K$.

As $\gamma^i$ links $K$, it is linked with a cycle $\Gamma^{n-i-1}$ of $K$. Since $D$ is u.l.i-c. for $0 \leq i \leq n-j-2$, there lies in $D \cdot S(P, \epsilon)$ a cycle $\Gamma^{n-i-1}$ approximating $\Gamma^{n-i-1}$ and linked with $\gamma^i$.

THEOREM 2. Let the compact point set $M$ be the common boundary of (at least) two domains $D_1$ and $D_2$ such that $D_k$, $(k = 1, 2)$, is u.l.i-c. for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 3$. Also, let $(n_1+1)$-cycles of $D_1$ of diameter less than some fixed positive number $\theta$ bound in $D_1$. Then $M$ is a g.c. $(n-1)$-m..§

PROOF. CASE 1. Suppose $n_1 \geq n_2$. By Theorem 3 of G.C.M., $M$ is completely $i$-avoidable at all points, for $0 \leq i \leq n_2$. We first prove that $D_1$ is u.l. $(n_1+1)$-c. If $D_1$ is not u.l. $(n_1+1)$-c., there exist $P \subset M$ and $\epsilon > 0$ such that for each $\eta > 0$ there exists a

* Theorem 1 is a generalization of Theorem 4 of my paper Concerning a problem of K. Borsuk, Fundamenta Mathematica, vol. 21 (1933), pp. 156-167. It should be noted that the neighborhoods $S(P, \epsilon)$ are relative to $E_n$, not merely to $M$.

† If $L$ is a chain, by $|L|$ we denote the set of points on $L$.

‡ See Lemma 2a of G.C.M. (A typographical error occurs in the statement of the lemma; the last "for" should be "1".)

§ Theorem 2 is an exact generalization of Theorem 8 of the paper in Fundamenta Mathematica, vol. 21, cited above.
cycle $\gamma^{n+1}$ in $D_1 \cdot S(P, \eta)$ that does not bound in $D_1 \cdot S(P, \epsilon)$. However, let us choose $\delta$ and $\eta$ to satisfy the complete $i$-avoidability requirement with $\eta < \theta$. By hypothesis, there exists $K_1\gamma^{n+2} \rightarrow \gamma^{n+1}$, in $D_1$ and hence (for $i = n_2$) in $E_n - H$ ($H$ as defined in proof of Theorem 1). Any $K_2\gamma^{n+2} \rightarrow \gamma^{n+1}$ in $S(P, \eta)$ also lies in $E_n - [F(P, \epsilon) + M - H]$. Then $K_1\gamma^{n+2} - K_2\gamma^{n+2}$ must link a cycle of $M \cdot F(P, \delta)$, else $\gamma^{n+1}$ bounds in $D_1 \cdot S(P, \epsilon)$. But then it is linked with a $\Gamma^m$ of $M \cdot F(P, \delta)$, where $M = n - (n_1 + 2) - 1 = n_2$; such a cycle, however, bounds on $M \cdot [S(P, \epsilon) - S(P, \eta)]$, hence in $E_n - (K_1\gamma^{n+2} - K_2\gamma^{n+2})$. Thus the supposition that $\gamma^{n+1}$ does not bound in $D_1 \cdot S(P, \epsilon)$ leads to a contradiction.

We may now show that $D_1$ is u.l.i-c. for $n_1 + 2 \leq i \leq n - 2$. Let $j$ be such a fixed value of $i$; we note that $n_2 \geq n - j - 1 \geq 1$. Suppose $D_1$ not u.l.i-c. Then we may determine a point $P$ of $M$ and an $\epsilon > 0$ such that for each $\eta > 0$ there is a cycle $\gamma^j$ of $D_1 \cdot S(P, \eta)$ that fails to bound in $D_1 \cdot S(P, \epsilon)$. Let $\delta$ and $\eta$ be such that (1) $\epsilon > \delta > \eta > 0$, (2) any $(n - j - 2)$-cycle of $M \cdot S(P, \delta)$ bounds in $M \cdot [S(P, \epsilon) - S(P, \eta)]$, (3) any $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$ bounds in $D_2 \cdot S(P, \epsilon)$ and hence in $D_2$, and (4) if an $\gamma^j$ links $M$, then (Theorem 1) it is linked with an $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$. Now if an $\gamma^j$ of $D_1$ were linked with $M$, we could by condition (4) determine an $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$ with which $\gamma^j$ is linked. As this would not be possible by condition (3), we can suppose that $\gamma^j$ does not link $M$. Then there exists a chain $K_1\gamma^{i+1} \rightarrow \gamma^j$ in $E_n - M$, hence in $E_n - M \cdot S(P, \delta)$. Let $K_2\gamma^{i+1}$ be an arbitrary chain of $S(P, \eta)$ bounded by $\gamma^j$, and we have $K_2\gamma^{i+1} \rightarrow \gamma^j$ in $E_n - [F(P, \epsilon) + M - M \cdot S(P, \delta)]$. As before, we see by applying condition (2) that $\gamma^j$ bounds in $D_1 \cdot S(P, \epsilon)$.

Thus $D_1$ is u.l.i-c. for $0 \leq i \leq n - 2$, and for this case the theorem follows from Principal Theorem C of G.C.M.*

**Case 2.** Suppose $n_1 < n_2$. In this case we show that $D_2$ is u.l.i-c. for $n_2 + 1 \leq i \leq n - 2$. We note that $M$ is completely $(n - j - 2)$-avoidable for $0 \leq n - j - 2 \leq n_1$ at all points. The proof then follows the general method of Case 1.

The following corollary is obvious.

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* That $D_1$ is simply $(n - 1)$-connected follows from the fact that $M$, being a common boundary of two domains, is a continuum.
Corollary. A compact set that is the common boundary of (at least) two domains $D_1$ and $D_2$ such that $D_k$, $(k = 1, 2)$, is u.l.i.c. for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 2$, is a g.c.(n-1)-m.

Theorem 3. Let $M$ be a common boundary of (at least) two domains $D_1$ and $D_2$ such that $D_k$, $(k = 1, 2)$, is u.l.i.c. for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 3$. Then if $p^{n_k+1}(D_k)$ is finite for either $k = 1$ or $2$, there exists a $\theta > 0$ such that $(n_k+1)$-cycles of $D_k$ of diameter $< \theta$ bound in $D_k$.*

Proof. Take, for instance, $p^{n_1+1}(D_1)$ finite. Let $n_1 + 1 = k$. Denote the cycles of a $k$-basis of $D_1$ by $\Gamma^k_i, (i = 1, 2, \ldots, m)$. By the method of proof of Theorem 5 of G.C.M. we can prove the following lemma.

Lemma. Let $D$ be a u.l.i.c. domain, $(0 \leq i \leq j)$, and let $\Gamma^k_i, (i = 1, 2, \ldots, m; 0 \leq n - k - 1 \leq j + 1)$, be a set of independent cycles linking $D$. Then in $D$ there exist independent cycles $\gamma^{n-k-1}_i, (i = 1, 2, \ldots, m)$, such that every linear combination of the $\Gamma$'s is linked with at least one $\gamma$.

Applying the lemma, we see that there exists in $D_2$ a set of $(n-k-1)$-cycles $\gamma^{n-k-1}_i, (i = 1, 2, \ldots, m)$, such that every linear combination of the $\Gamma$'s is linked with at least one of the $\gamma^{n-k-1}_i$. The remainder of the proof is similar to that for Theorem 14 of D.B. From Theorems 2 and 3 we have our principal result.

Principal Theorem. Let a compact point set $M$ be a common boundary of (at least) two domains $D_1$ and $D_2$ such that $D_k$, $(k = 1, 2)$, is u.l.i.c. for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 3$. Then, if one of the numbers $p^{n_k+1}(D_k)$ is finite, $M$ is a g.c.(n-1)-m.

For the case $n = 3$, where necessarily the numbers $n_1$ and $n_2$ as defined above must equal 0, I have shown in D.B. that without the single condition as to the finiteness of one of the numbers $p^{n_k+1}(D_k)$, not only may the boundary fail to be a manifold, but it may be the common boundary of three or more domains. However, if $M$ has a single point $P$ such that all 1-cycles of $D_k \cdot S(P, \epsilon)$ bound in $D_k$, then $M$ is the common boundary

* Compare Theorem 14 of D.B.
of only two domains. (Theorem 11 of D.B.) We now extend the latter result to higher dimensions.*

**Theorem 4.** Let $M$ be a common boundary of two domains $D_1$ and $D_2$ such that $D_k$, $(k = 1, 2)$, is u.i.-c. for $0 \leq i \leq n_k$, where $n_1 + n_2 = n - 3$. Then, if for (at least) one of the values of $k$, there exists a point $P$ of $M$ and an $\epsilon > 0$ such that all $(n_k + 1)$-cycles of $D_k \cdot S(P, \epsilon)$ bound in $D_k$, it follows that $M$ is the common boundary of only two domains. Indeed, at $P$, $M$ is locally a g.c.$(n-1)$-m.†

**Proof.** Let $n_1 \geq n_2$. As both $D_1$ and $D_2$ are u.i.0-c., $M$ is a Jordan (or Peano) continuum, and the component $C$ of $M \cdot S(P, \epsilon)$ determined by $P$ is an open subset of $M$. By the method of argument used for Theorem 9 of D.B., $C$ is the common boundary of two u.i.-c., $(0 \leq i \leq n_k)$, domains $D_k'$, $(k = 1, 2)$, in $S(P, \epsilon)$, where all points of $D_k'$ in a certain neighborhood $U$ (rel. $E_n$) of $C$ belong to $D_k$ and conversely. As in Theorem 3 of G.C.M. we show that $C$ is completely $i$-avoidable at all points for $0 \leq i \leq n_2$.

We may now proceed, as in Theorem 2 above, to show that one of the domains $D_k'$ is u.i.-c. for $0 \leq i \leq n - 2$ at all points of $C$. Following this, we may show by methods such as those used to prove Theorem 12 of G.C.M. that in $U$ there exist only points of $C + D_1 + D_2$.

In conclusion we note that in higher dimensions there exist, a priori, further possibilities concerning common boundaries of several domains. For instance, does there exist for some $E_n$ a common boundary of three domains $D_k$, $(k = 1, 2, 3)$, such that $D_k$ is u.i.-c. for $0 \leq i \leq n_k$, where $n_1 > n_2 > n_3$? The answer, in case $n_1 + n_3 \geq n - 3$, is clearly negative by virtue of the corollary to Theorem 2 above; and indeed we must have $n_1 + n_2 \leq n - 3$ in such a case. For the case $n_1 + n_2 = n - 3$, let us consider the Betti numbers $p^{n+m}(E_n - M)$, where $n_1 + m \leq n - 2$ and $n - (n_1 + m) - 1 \leq n_3$ (if any such exist). By the proof of Theorem 4 of G.C.M. we may show $p^i(B)$ finite for $0 \leq i \leq n_3$. Consequently the num-

* It will be noted that we show now that the “$\epsilon$-condition” is needed only for one domain.

† That is, conditions (2), (3) of definition $M^{n-1}$ of G.C.M. are satisfied for some connected open neighborhood $U$ of $P$, and so on.
bers $p^{n+m}(E_n-M)$ are all finite. Thus we have the following theorem.

**Theorem 5.** Let $M$ be a common boundary of three distinct domains $D_k$, $(k=1, 2, 3)$, such that $D_k$ is u.l.i-c. for $0 \leq i \leq n_k$, and $n_1 \geq n_2 \geq n_3$. Then $n_1 + n_2 \leq n - 3$, and if there exists $m > 0$ such that $n_1 + m \leq n - 2$ and $n - (n_1 + m) - 1 \leq n_3$, the Betti numbers $p^{n+m}(E_n-B)$ and $p^i(B)$, $(0 \leq i \leq n_3)$, are all finite.*

* Thus, although we have no actual example, it is conceivable that there exists, in $E_3$, a common boundary $M$ of three domains $D_k$ each of which is u.l.i-c. for $i=0, 1$. If so, $p^i(D_k)$ is infinite for $k=1, 2, 3$; and $p^i(E_3-M)$ is finite.

† Grace and Young, *The Algebra of Invariants*, 1903, Chapter 11.