DERIVED SETS AND THEIR COMPLEMENTS

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1. *Introduction.* Denoting by gA the derived set of a set A in a general topological space* S, and by cA the complement $S-A$ of A, we consider the family of sets ϕA, where ϕ is a product involving the operator c and operators of the form g^α, α being an ordinal, finite or transfinite. Important examples of operators are cg, gc, g^2c, cgc. The following discussion is based on the assumption of the distributive and closure properties:

I. $$g(A + B) = gA + gB,$$

II. $$g^2A \subset gA.$$

In §§2–5 eight *elementary* sets are defined, from which a *canonical* system of sets is obtained. This canonical system is sufficient for the representation of all sets of the form ϕA and the finite sums and products $\sum\phi A$, $\prod\phi A$, and $\sum\prod\phi A$, with the exception of certain subsets of the derived set of the isolated points of the space S. In §6 specializations of the general theory are given to spaces possessing either or both of the properties: (a) self density, (b) $g0=0$. Under restrictions (a) and (b), the basic set of inclusions (10), which is fundamental for the discussion of §§4 and 5, is found to be logically equivalent to a set of inclusions given by Kuratowski† for the set $\overline{A}$, having the properties of the closure $A+gA$ of A. In §7 are presented various properties of the elementary sets. All relations are established formally, though Axioms I, II are equivalent to the assumption of a neighborhood space with open sets for neighborhoods, so that all results may be had by classification of neighborhoods with respect to the distribution of the points of A and cA. The symbols $\rightarrow$ and $\subset$ denote respectively implication and inclusion.

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* See M. Fréchet, *Les Espaces Abstraits*, 1928; E. W. Chittenden, *On general topology*, Transactions of this Society, vol. 31 (1929), pp. 290–321; W. Sierpinski, *La notion de derivée comme base d'une théorie des ensembles abstraits*, Mathematische Annalen, vol. 97 (1926), pp. 321–337.

† C. Kuratowski, *Sur l'opération $\overline{A}$ de l'analysis situs*, Fundamenta Mathematica, vol. 3 (1922), pp. 182–199.

2. *Reduction Formulas*. From the distributive law I we have $gA + gcA = gS$. Taking complements, we obtain the following lemma.

LEMMA 1. *We have* $cgA \subset gcA + J$, *where* $J = cgS$, *the set of isolated points of the space.*

From I also follows the monotonic property

$$X \subset Y \rightarrow gX \subset gY. \tag{1}$$

Since $X \subset Y \rightarrow cY \subset cX$, we have also, by (1),

$$\begin{aligned} X \subset Y &\rightarrow gcY \subset gcX, \\ &\rightarrow cgY \subset cgX, \\ &\rightarrow gcgY \subset gcgX. \end{aligned}$$

Instead of referring to the monotonic property (1), it will be frequently convenient to speak of *operating with* g *on* $X \subset Y$. Likewise we shall operate with c, $gc, \cdots$.

LEMMA 2. *For any operator* ϕ, $J \subset cg\phi A$.

The lemma follows on applying the operator cg to the inclusion $\phi A \subset S$.

LEMMA 3. *For any operator* ϕ, $gJ \subset gcg\phi A$.

This is an immediate consequence of (1) and Lemma 2.

The following reduction formulas hold for any ordinal α.

FORMULA I. $gcgA = gcg^{\alpha}A$. Operating with g on Lemma 1, we obtain by the closure property II,

$$gcgA \subset g^2cA + gJ \subset gcA + gJ, \tag{2}$$

whence, on replacing A by gA, we find $gcg^2A \subset gcgA + gJ$; or by Lemma 3, $gcg^2A \subset gcgA$. The reversed inclusion is obtained by operating with gc on the inclusion $g^2A \subset gA$, so that we shall have $gcgA = gcg^2A$. Replacing A by gA, we get $gcgA = gcg^3A$, and the desired relation follows at once.

FORMULA II. $gcgA = g^{\alpha}cgA + gJ$. Replacing A by cgA in $g^2A \subset gA$, we shall have $g^2cgA \subset gcgA$, whence, by Lemma 3,

$$g^2cgA + gJ \subset gcgA. \tag{3}$$

Replacing A by gA in (2), we get $gcg^2A \subset g^2cgA + gJ$, whence, by Formula I, $gcgA \subset g^2cgA + gJ$. From this and (3) follows

$$(4) \qquad gcgA = g^2cgA + gJ.$$

Operating on (4) with g, we have $g^2cgA = g^3cgA + g^2J$, whence (4) may be written in the form $gcgA = g^3cgA + gJ$. The required formula follows immediately.

Preliminary to Formula III we have the following lemma.

LEMMA 4. $gXcgcY \subset g(XY) \subset gXgY$.

The second inclusion is a direct consequence of the monotonic property. As for the first, we may write, by I,

$$gX = g(XY + XcY) = g(XY) + g(XcY),$$

whence

$$(5) \qquad gXcgcY = [g(XY) + g(XcY)]cgcY \subset g(XY) + g(XcY)cgcY;$$

but, by the second inclusion of the lemma, $g(XcY) \subset gXgcY$, so that we have $g(XcY)cgcY \subset gXgcYcgcY = 0$, and, by (5), the lemma is established.

FORMULA III. $gcgA = gcgcgcgA$. In Lemma 1 we replace A by cgA, obtaining, by II, $cgcgA \subset g^2A + J \subset gA + J$, whence, on operating by g,

$$(6) \qquad gcgcgA \subset g^2A + gJ \subset gA + gJ.$$

Again, replacing A by cgA, we have, by Lemma 3,

$$gcgcgcgA \subset gcgA + gJ = gcgA.$$

To establish the reversed inclusion, we proceed as follows. Operating on (6) with gc, we have

$$(7) \qquad g(cgA\,cgJ) \subset gcgcgcgA.$$

Now in Lemma 4, on setting $X = cgA$, $Y = cgJ$, we shall have $gcgA\,cg^2J \subset g(cgA\,cgJ)$, or by the closure property,

$$(8) \qquad gcgA \subset g(cgA\,cgJ) + gJ.$$

But, by Lemma 3, $gJ \subset g(cgA\,cgJ)$, so that (8) reduces to

$$(9) \qquad gcgA \subset g(cgA\,cgJ).$$

By (7) and (9) the desired relation is established.

3. *A Canonical System of Sets.* Consider now a set ϕA, where ϕ is any combination of the deriving and complementing operations. By Formulas I, III, we have

$$g^\alpha cg^\beta cg^\gamma cg^\delta A = g^\alpha cgcgcgA = g^\alpha cgA.$$

Thus it appears that when any derived set of A is operated on with gc, the only distinct sets obtainable by further operations with g and c are of the types $g^\alpha cgA$, $cg^\alpha cgA$, $g^\alpha cgcgA$, $cg^\alpha cgcgA$. The following theorem is thus established.

THEOREM 1. *For any operator ϕ, the set ϕA reduces to A, cA, cgA, $cgcA$, $cgcgA$, $cgcgcA$, or else may be obtained by operating on one of these sets with g^α or cg^α.*

4. *Definition of the Elementary Sets.* The following lemma is needed.

LEMMA 5. *For any ordinals α, β, and operator ϕ,*

$$g^\alpha cg\phi A = g^\beta cg\phi A cgJ + g^\alpha cg\phi A gJ,$$
$$cg^\alpha cg\phi A = cg^\beta cg\phi A cgJ + cg^\alpha cg\phi A gJ.$$

By the closure property and the Formula II previously stated, we have

$$g^\alpha cg\phi A \subset gcg\phi A = g^\beta cg\phi A + gJ,$$

whence we obtain $g^\alpha cg\phi A cgJ = g^\beta cg\phi A cgJ$, and

$$g^\alpha cg\phi A = g^\alpha cg\phi A cgJ + g^\alpha cg\phi A gJ = g^\beta cg\phi A cgJ + g^\alpha cg\phi A gJ.$$

The rest of the lemma follows on taking complements.

The following fundamental inclusions hold for all values of α.

I1. $$cg^\alpha cA c\bar{J} \subset cgcgcgcA cJ.$$

Operating on (6) with $cg^{\alpha-1}$, and replacing A by cA, we have $cg^\alpha cA cgJ \subset cg^\alpha cgcgcA$, whence $cg^\alpha cA cgJ \subset cg^\alpha cgcgcA cgJ$. But, choosing $\beta = 1$ in Lemma 5, we have $cg^\alpha cgcgcA cgJ = cgcgcgcA cgJ$, so that the preceding inclusion becomes, by Lemma 3,

$$cg^\alpha cA cgJ \subset cgcgcgcA cgJ = cgcgcgcA.$$

Multiplying by cJ, we have $cg^\alpha cA c\bar{J} = cg^\alpha cA cgJ \subset cgcgcgcA cJ$, which is the desired inclusion.

I2. $$cgcgcgcA cJ \subset g^\alpha cgcA.$$

In Lemma 1 we replace A by $cgcgcA$, and have, by II,

$$cgcgcgcA \subset g^2cgcA + J \subset gcgcA + J.$$

Multiplying by $cgJcJ$, we obtain $cgcgcgcAcJ \subset gcgcAcgJcJ$, by Lemma 3. But by Formula II, $gcgcAcgJ = g^\alpha cgcAcgJ$, whence the above inclusion yields $cgcgcgcAcJ \subset g^\alpha cgcAcgJcJ \subset g^\alpha cgcA$.

I3. $$cgcgcgcAcJ \subset cg^\alpha cgA.$$

Applying the operator gc to (2), we get $g(cgcAcgJ) \subset gcgcgA$. Since, by (9), $gcgcA \subset g(cgcAcgJ)$, we have then $gcgcA \subset gcgcgA$, or, replacing A by cA, $gcgA \subset gcgcgcA \subset gcgcgcA + J$. From the closure of derived sets, we have $g^\alpha cgA \subset gcgA \subset gcgcgcA + J$. On taking complements, the desired inclusion is obtained.

By operating with c on I3, I2, I1, respectively, and replacing A by cA, the following inclusions are obtained:

I4. $$g^\alpha cgcA \subset gcgcgA + J.$$

I5. $$cg^\alpha cgA \subset gcgcgA + J.$$

I6. $$gcgcgA + J \subset g^\alpha A + \bar{J}.$$

Consider now the following special forms of the inclusions I1–I6, where α_1 and α_2 are the least ordinals α such that $g^\alpha cgcA$ and $g^\alpha cgA$, respectively, are perfect:

$$(10)\qquad cgcAc\bar{J} \subset cgcgcgcAcJ \begin{matrix} \subset g^{\alpha_1}cgcA \subset \\ \subset cg^{\alpha_2}cgA \subset \end{matrix} gcgcgA + J \subset gA + \bar{J}.$$

From the monotonic nature of the six sets (10), it is seen that the first, the complement of the sixth, and the differences of the sixth and fifth, the fifth and second, and the second and first, are mutually disjoined and fill the space S. The difference of the fifth and second sets, namely, $(gcgcgA+J)$ $(gcgcgcA+J)$, may be decomposed into four non-overlapping sets by means of the intervening sets, $g^{\alpha_1}cgcA$ and $cg^{\alpha_2}cgA$. Thus we obtain a decomposition of S into eight non-overlapping sets. With certain reductions by (10), Lemmas 2 and 3, we list these eight sets, and shall refer to them as elementary or, more briefly, E sets:

$$E_1 = cgcAc\bar{J}, \qquad E_2 = cgcgcgcAcJc(cgcAc\bar{J}) = cgcgcgcAgcA,$$
$$E_3 = g^{\alpha_1}cgcAcg^{\alpha_2}cgAc(cgcgcgcAcJ) = g^{\alpha_1}cgcAcg^{\alpha_2}cgAgcgcgcA,$$
$$(11)\quad E_4 = cgAc\bar{J}, \qquad E_5 = (gA + \bar{J})c(gcgcgA + J) = gAcgcgcgA,$$
$$E_6 = (gcgcgA + J)cg^{\alpha_1}cgcAc(cg^{\alpha_2}cgA) = gcgcgAcg^{\alpha_1}cgcAg^{\alpha_2}cgA$$
$$E_7 = g^{\alpha_1}cgcAg^{\alpha_2}cgA, \qquad E_8 = cg^{\alpha_1}cgcAcg^{\alpha_2}cgA.$$

It is seen that the replacement of A by cA interchanges E_1 and E_4, E_2 and E_5, E_3 and E_6. Thus, corresponding to a theorem holding for a set of one of these pairs, there is an identical theorem valid for the other set. We shall make use of this symmetry.

5. *The General Canonical System.*

LEMMA 6. $J \subset E_8$; $gJ \subset E_3+E_6+E_7+E_8$.

It is seen by (11) that every E set except E_8 has either the factor cJ or else a factor $g\phi A$, and so by Lemma 2 is disjoined from J. Likewise E_1 and E_4 contain the explicit factor cgJ, while E_2 and E_5 have factors $cgcgcgcA$ and $cgcgcgA$, which by Lemma 3 are contained in cgJ.

By Lemma 6, 3, and 5, we may obtain from (11) the following decomposition of the sets mentioned in Theorem 1.

$$
(12)\quad
\begin{aligned}
g^\alpha cgA &= E_4+E_5+cgJ(E_6+E_7)+g^\alpha cgA\,gJ.\\
cg^\alpha cgA &= E_1+E_2+cgJ(E_3+E_8)+cg^\alpha cgA\,gJ.\\
g^\alpha cgcA &= E_1+E_2+cgJ(E_3+E_7)+g^\alpha cgcA\,gJ.\\
cg^\alpha cgcA &= E_4+E_5+cgJ(E_6+E_8)+cg^\alpha cgcA\,gJ.\\
g^\alpha cgcgA &= E_1+E_2+cgJ(E_3+E_6+E_7)+c\bar{J}E_8+g^\alpha cgcgA\,gJ.\\
cg^\alpha cgcgA &= E_4+E_5+J+cg^\alpha cgcgA\,gJ.\\
g^\alpha cgcgcA &= E_4+E_5+cgJ(E_3+E_6+E_7)+c\bar{J}E_8+g^\alpha cgcgcA\,gJ.\\
cg^\alpha cgcgcA &= E_1+E_2+J+cg^\alpha cgcgcA\,gJ.
\end{aligned}
$$

Expressions for the sets $g^\alpha A$, $cg^\alpha A$, $g^\alpha cA$, $cg^\alpha cA$ of Theorem 1 in terms of the E sets cannot be obtained in this manner, since Lemma 5 does not apply to these sets. We have instead the following lemma.

LEMMA 7.

$$P_1cgJ = E_1 + E_2 + cgJ(E_3 + E_6 + E_7) + c\bar{J}E_8 + P_1E_5,$$

$$P_2cgJ = P_2E_2 + cgJE_3 + E_4 + E_5 + cgJ(E_6 + E_7) + c\bar{J}E_8,$$

where P_1, P_2, *are respectively the perfect components of* A, cA.

By (11), Lemma 6, and I6, we have for every α

$$(13)\quad E_1 + E_2 + cgJ(E_3 + E_6 + E_7) + c\bar{J}E_8 = gcgcgA\,c\bar{J} \subset g^\alpha A.$$

Since E_4 is disjoined from gA, the lemma follows.

By (11) and Lemma 7 we have

$$
(14)\quad
\begin{aligned}
g^{\alpha}A &= E_1 + E_2 + cgJE_3 + P_1E_5 + \textstyle\sum_{\beta\geqq\alpha} (g^{\beta}A - g^{\beta+1}A) \\
&\qquad + cgJ(E_6 + E_7) + c\bar{J}E_8 + P_1gJ, \\
cg^{\alpha}A &= E_4 + \textstyle\sum_{\beta<\alpha} (g^{\beta}A - g^{\beta+1}A) + J + gJcg^{\alpha}A, \\
g^{\alpha}cA &= P_2E_2 + \textstyle\sum_{\beta\geqq\alpha} (g^{\beta}cA - g^{\beta+1}cA) + cgJE_3 + E_4 + E_5 \\
&\qquad + cgJ(E_6 + E_7) + c\bar{J}E_8 + P_2gJ, \\
cg^{\alpha}cA &= E_1 + \textstyle\sum_{\beta<\alpha} (g^{\beta}cA - g^{\beta+1}cA) + J + gJcg^{\alpha}cA.
\end{aligned}
$$

In (12) and (14) we have the required decomposition of the sets of Theorem 1. Since by (13) each set $g^{\beta}A - g^{\beta+1}A$ lies in $E_5 + gJ$, we may sum up our results in the following theorem.

THEOREM 2. *Any finite sum of finite products* $\Sigma\Pi\phi A$ *is expressible, aside from a subset of* gJ, *as a sum chosen from* E_1, P_2E_2, $g^{\beta}cA - g^{\beta+1}cA$, $cgJE_3$, E_4, P_1E_5, $g^{\beta}A - g^{\beta+1}A$, $cgJE_6$, $cgJE_7$, $c\bar{J}E_8$, J, *and products of these sets by* A *and* cA.

To illustrate, consider the set $X = g^2cgcgcgA\,cgcg^2cgA\,g^3A\,gcgA$. Applying Formulas I and III, we have $X = g^2cgA\,cgcgcgA\,g^3A\,gcgA$, whence by the closure of derived sets, $X = g^2cgA\,cgcgcgA\,g^3A$. Finally, (12) and (14) give $X = P_1E_5 + \sum_{\beta\geqq 3}(g^{\beta}A - g^{\beta+1}A)$, aside from a subset of gJ.

6. *Specializations.* Most of the spaces commonly occurring in geometry and analysis, such as Euclidean n-space, Hilbert space, and all continua, have the self-dense property, and also the property $g0 = 0$. If we set $J = gJ = 0$, the results of the preceding sections are considerably simplified. Formula II reduces to $gcgA = g^{\alpha}cgA$, and the inclusions (10) become*

$$
(15)\qquad cgcA \subset cgcgcgcA \begin{matrix} \subset gcgcA \subset \\ \subset cgcgA \subset \end{matrix} gcgcgA \subset gA.
$$

The elementary sets (11) reduce to

$$
(16)\quad
\begin{aligned}
&E_1 = cgcA, \quad E_2 = cgcgcgcA\,gcA, \quad E_3 = gcgcgcA\,gcgcA\,cgcgA, \\
&E_4 = cgA, \quad E_5 = gA\,cgcgcgA, \quad E_6 = gcgcgA\,gcgA\,cgcgcA, \\
&E_7 = gcgcA\,gcgA, \quad E_8 = cgcgcA\,cgcgA,
\end{aligned}
$$

* These inclusions are given by Kuratowski, loc. cit., for the closure function.

while the general canonical system of §5 reduces to

$$(17)\quad \begin{array}{l} E_1, \quad P_2E_2, \quad \sum (g^\beta cA - g^{\beta+1}cA), \quad E_3, \quad E_4, \quad P_1E_5, \\ \qquad\qquad\qquad \sum (g^\beta A - g^{\beta+1}A), \quad E_6, \quad E_7, \quad E_8. \end{array}$$

The requirement $gJ=0$ yields a space slightly more general than the preceding, including the case where isolated points J exist, yet have a null derived set. The formulas of §§2–5 specialize, however, to the very forms (15), (16), (17). The structure of the E sets is unchanged except that E_8 includes the set J.

We may also specialize the formulas of §§2–5 by requiring $J=0$, thus including the case $J=gJ=0$ and also the case where the derived set of the null set is not null. By the monotonic property we have for every set A, $g0=gJ\subset gA$, so that the inclusions (10) again reduce to (15), and the definitions (16) hold without change. Since $gJ\subset gA$, we see that E_7 includes gJ.

If we require $J+gJ=S$, the canonical system of Theorem 2 is considerably simplified, since the inclusions (10) reduce to

$$0 \begin{array}{c} \subset g^{\alpha_1}cgcA \subset \\ \subset cg^{\alpha_2}cgA \subset \end{array} S,$$

so that the only non-null E sets are E_3, E_6, E_7, E_8.

7. *The Nature of the Elementary Sets*. The following theorems may be obtained.

THEOREM 3. *$E_1+E_4+E_8+gJ$ is dense on gS.*

THEOREM 4. $E_3cgJ=gE_1cgE_4gE_8cgJ$.

THEOREM 5. $E_2=cgJgE_1cgE_4cgE_8gcA$.

THEOREM 6. $gE_3\subset E_3+E_7$.

THEOREM 7. $gE_2\subset E_2+E_3+E_7+gJ$.

THEOREM 8. $gE_8\subset E_3+E_6+E_7+E_8$.

In a space $J=gJ=0$ it is evident that $E_8=cgE_1cgE_4$, and $E_7=gE_1gE_4$. If the space is also of n dimensions, then it follows from Theorem 3 that E_1, E_4, and E_8 are n-dimensional or else null, whereas the remaining E sets are at most $(n-1)$-dimensional. Specializing further, we have the following result.

THEOREM 9. *In a Euclidean space of more than one dimension, $E_7\subset gE_7+gE_3gE_6$.*

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