THE VENERONI TRANSFORMATION IN $S_n^*$

BY VIRGIL SNYDER AND EVELYN CARROLL-RUSK

1. Introduction. The purpose of the present paper is to obtain the system of bilinear equations of the Veneroni transformation defined by associating projectively the primes (or hyperplanes) of $S_n$ with the primais (or hypersurfaces) $V^n$ of order $n$ through $n+1$ arbitrary $S_{n-2}$ of $S_n'$, and to derive a complete scheme of mapping of the manifolds of either space on the linear manifolds of the other. Both of these are believed to be new.

2. Analytic Expression for the Veneroni Transformation. The process is somewhat different according as $n$ is odd or even. It will be most easily understood by considering in detail the case $n=5$. In $S_5$ six three way spaces $\sigma_i$, $i=1, 2, \cdots, 6$, form the base of a Veneroni transformation. Let $\sigma_i\equiv x_i=0, x_2=0; \sigma_2\equiv x_2=0, x_4=0, \sigma_3\equiv x_t=0, x_6=0, \Sigma^6_{i=1}a_ix_i=0, b=0; \sigma_5\equiv c=0, d=0; \sigma_6\equiv e=0, f=0$. Through five of the $\sigma_k$ (not $\sigma_k$) passes one and only one $V_5^A$, say $V_5$, determined by the $\infty^3$ lines meeting all of them. This and any $S_4$ of the pencil through the remaining $\sigma_k$ provides one composite quintic primal of the system. Thus a complete homaloidal system is obtained. Among these primais six independent identities exist of the form

$$x_2V_1 = (e,f)V_6 + (c,d)V_6 + (a,b)V_4,$$

in which $(a,b)=a_ib(x)-b_ia(x), \cdots$. If $\rho x_1'=x_1V_1, \rho x_2'=x_2V_1, \rho x_3'=x_3V_2, \cdots$, we may write

$$\frac{a(x)}{b(x)} = -\frac{\sum B_ix_i'}{\sum A_ix_i'},$$

and so on where $B_i$ is the cofactor of $b$, in the determinant

$$D = \begin{vmatrix} a_1b_2c_3d_4e_5f_6 \end{vmatrix},$$

after the transpositions (12), (34), (56) have been made in the subscripts. Any two of these and the three equations

$$x_1x_4' - x_1'x_2 = 0, x_3x_4' - x_3'x_4 = 0, x_5x_6' - x_5'x_6 = 0$$

* Presented to the Society, September 11, 1935.
completely define the transformation. A similar statement holds for \( n \) any odd number. For \( n \), even a slight variation in the procedure must be introduced, since the equations of the \( \sigma_i \) can not all be taken as part of the simplex of reference.

3. Double Elements of \( V_{n-1}^{n-1} \) in \( S_n \). Any two primals of \( S_{n-1} \) of the system, \( V_\alpha \) and \( V_\beta \), have in common all the \([n-2]\)-dimensional bases \( \sigma_i \) except \( \sigma_\alpha \) and \( \sigma_\beta \); a ruled manifold \( R_{n-3}^{(n+1)(n-2)/2} \) of order \((n+1)(n-2)/2\) consisting of the lines intersecting all the \( \sigma_i \); and a residual manifold \( M_{n-2}^{(n-2)(n-3)/2} \) of order \((n-2)(n-3)/2\).

The bases \( \sigma_i, \sigma_k \) meet in an \([n-4]\)-space denoted by \( \sigma_{i,k} \). The spaces \( \sigma_{12}, \sigma_{13}, \cdots, \sigma_{1(n+1)} \) are double and lie in \( \sigma_1 \). On an \([n-1]\)-primal, such as \( V_{n+1}^{n+1} \), the bases \( \sigma_{12}, \sigma_{13}, \cdots, \sigma_{1n} \) lie in an \( S_{n-2} \) and are intersected by a ruled manifold \( R_{n-4}^{(n-1)(n-4)/2} \). These \([n-4]\)-spaces \( \sigma_{12}, \sigma_{13}, \cdots, \sigma_{1n} \) and the generators of the ruled manifold are double on \( V_{n+1} \) and form the base of a homaloidal system of primals in \( S_{n-1} \) in \( \sigma_1 \). The manifolds of this system which are cut by \( V_\alpha \) and \( V_\beta \) intersect therefore in a normal variety \( M_{n-1}^{(n-1)(n-2)/2} \). In the general case, there are no other double elements. Similarly, the presence of triple and other multiple loci on each \( V_{n-1}^{n-1} \) of the system can be obtained.

For \( n=5 \), on each \( V_4^{n-1} \) containing five base three spaces \( \sigma_i \) there are twenty double lines and five double space cubic curves, all lying in the base spaces \( \sigma_i \). These results were obtained by Eiesland by the method of differential geometry.

4. Residual Intersection of Two \( V_{n-1}^{n-1} \) in \( S_n \). The intersection of \( V_\alpha \) and \( V_\beta \), which is of order \((n-1)^2\), consists of \( n-1 \) of the bases \( \sigma_i \), of the ruled manifold \( R_{n-2}^{(n+1)(n-2)/2} \), and of a residual manifold of order \((n-2)(n-3)/2\).

Each \( S_{n-1} \) of the pencil \( |\sigma_k|, (k \neq \alpha, \beta) \), meets each of the \( n-1 \) remaining \( \sigma_i \), \((i \neq \alpha, \beta)\), in an \( S_{n-3} \). These \( S_{n-3} \) intersect in pairs in \([n-5]\)-spaces \( S_{n-5} \), belonging to a common \([n-3]\)-space \( K_{n-3} \) which meets each \( S_{n-3} \) in an \( S_{n-4} \). This \( K_{n-3} \) is incident to the \( \sigma_k \) and meets \( \sigma_\alpha \) and \( \sigma_\beta \). Hence it lies on \( V_\alpha \) and \( V_\beta \) and, consequently, on each primal of the pencil. As \( S_{n-1} \) describes the pencil \( |\sigma_k| \), this \( K_{n-3} \) describes the variety \( M_{n-2}^{(n-2)(n-3)/2} \). Its equations are

\[* J. Eiesland, Palermo Rendiconti, vol. 54 (1930), pp. 335–365.\]
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\[
\begin{align*}
\frac{x_1}{x_2} &= \frac{(a_2 b_3) x_3 + (a_4 b_2) x_4}{(a_1 b_3) x_3 - (a_4 b_1) x_4} = \frac{(a_5 b_2) x_5 + (a_6 b_2) x_6}{(a_1 b_5) x_5 - (a_6 b_1) x_6} = \ldots \\
&= \frac{(a_5 b_2) x_n + (a_{n+1} b_2) x_{n+1}}{(a_1 b_n) x_n - (a_{n+1} b_1) x_{n+1}}.
\end{align*}
\]

For \(n = 5\), if \(V_5\) and \(V_6\) are determined by \(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\) and \(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\), respectively, the intersection consists of the bases \(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\) of the three-dimensional variety \(R^3\), and of a residual variety of order three. Each of the \(S_4\) of the pencil \(|\sigma_4|\) meets \(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\), in the planes \(\pi_1, \pi_2, \pi_3\), respectively. These planes meet in pairs in the points \(\pi_{12}, \pi_{13}, \pi_{23}\) which determine a plane \(\pi_{123}\) meeting each \(\pi_i\) in a line. \(\pi_{123}\) also meets \(\sigma_4\) in a line and \(\sigma_5\) and \(\sigma_6\), each in a point. The plane \(\pi_{123}\) therefore lies on both \(V_5\) and \(V_6\). As the \(S_4\) describes the pencil, the plane \(\pi_{123}\) describes the cubic variety

\[
\begin{align*}
\frac{x_1}{x_2} &= \frac{(a_2 b_3) x_3 + (a_4 b_2) x_4}{(a_1 b_3) x_3 - (a_4 b_1) x_4} = \frac{(a_5 b_2) x_5 + (a_6 b_2) x_6}{(a_1 b_5) x_5 - (a_6 b_1) x_6}.
\end{align*}
\]

5. Transformations in \(S_n\) Defined by \(n\) Bilinear Equations. The \(n\) bilinear equations

\[
A_r \equiv \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} a_{i,k} x_i x_k' = 0, \quad (r = 1, 2, \ldots, n)
\]

define each \(x_i'\) as a function of \((x)\), expressed as a determinant of order \(n\), each element being linear. These determinants belong to a matrix \(\|a_{i,j}\|\) of \(n\) rows and \(n+1\) columns. They are satisfied by a manifold \(M\) of order \((n+1)n/2\) and of dimensionality \(n-2\). Thus, for \(n = 3\), there results a \(C_6\), defining a web of cubic surfaces. Conversely, the system of primals of order \(n\) containing \(M\) form a homaloidal system.

An \(S_{n-k}\), defined by \(k\) primes in \((x')\), is transformed into a variety of order \(n(n-1) \cdots (n-k+1)/k!\) and of dimensionality \(n-k\), having in common with \(M\) a variety of order \(k \cdot C_{k+1} \cdot (n+1)/(n-k)\) and of dimensionality \(n-k-1\).

Thus, if \(n = 3, k = 2\), a line of \((x')\) is transformed into a space cubic curve meeting \(C_6\) in eight points. The lines meeting \(M\) in \(n\) points generate a primal of order \(n^2-1\). No line has \(n+1\) points on \(M\) apart from the generators.*

In addition to the order \((n+1)n/2\) of \(M\) the orders of the double, triple, and other multiple manifolds on this given variety \(M\) can be expressed by equating to zero all the determinants of order \(n\), \(n-1\), \(\cdots\), respectively. The \(h\)th variety of this system exists if \(d-h(d-n+2h)\geq 0\), \(d\) being the dimensionality of the space, and this is its dimension. Its order is given by

\[
\frac{(d - n - 2q)h(d + n - 2q + 1)h \cdots (d + n - q)h}{(h)(h + 1)h \cdots (n)h},
\]

where \(n-h=q;^* (k)h\) represents the number of combinations of \(k\) things, taken \(h\) at a time.

6. Maps of \(S_1, S_2, \cdots \) in a General \((n, n)\) of Veneroni Type.

(a) If the point \(x\) describes a straight line, the image \(x'\) describes a normal \(C_n\) of \(S_n'\), as is seen by solving the equations of the line for every \(x_i\) in terms of two homogeneous ones, and then regarding these as the parameters in the locus of \(x'\). The curve may also be defined as the locus of intersections of corresponding primes of \(n-1\) projective pencils of primes.

(b) If the point \(x\) describes a plane, from its equations all the coordinates \(x_i\) can be eliminated except three, which appear linearly and homogeneously. The \(n\) defining bilinear equations can be interpreted as \(n\) projective systems of \(|S_{n-1}|\), each having an \(S_{n-3}\) for base.

For example, if the coordinates remaining in \((x)\) are replaced by \(\lambda, \mu, \nu\), there are \(n\) projective systems \(\lambda p^{(i)} + \mu q^{(i)} + \nu r^{(i)} = 0\), \(p^{(i)} = 0\), \(q^{(i)} = 0\), and \(r^{(i)} = 0\) being the equations of primes in \((x')\) for \(i = 1, 2, \cdots, n\). Each \(S_{n-1}\) defined by one of these equations passes through \(p^{(i)} = 0\), \(q^{(i)} = 0\), and \(r^{(i)} = 0\), hence through an \(S_{n-3}\). By eliminating \(\lambda, \mu, \nu\) there results a matrix of three rows and \(n\) columns, each element being linear in \(x'_i\). The locus common to all these is a surface \(F_2\) of order \(n(n-1)/2\). Each \(x'_i\) is a rational polynomial of order \(n\) in \(\lambda, \mu, \nu\). Since a variable double point can not exist in a linear system of plane curves by Bertini's theorem, the intersections of \(F_2\) and each surface \(S_{n-1}\) are represented by plane curves of order \(n\) with simple base points. Hence \(F_2\) contains \(n(n+1)/2\) lines, and the intersection of \(F_2\) and any \(S_{n-1}\) is of genus \((n-1)(n-2)/2\).

The \( n(n+1)/2 \) lines on \( F_2 \) are the images of the points in which a plane in \( (x') \) meets the \( n+1 \) base \([n-2]\)-spaces \( \sigma'_i \) and of those in which it meets the ruled manifold \( R^{(n+1)(n-2)/2} \). The images of the first set of points are the lines meeting \( n \) of the base \( \sigma_i \), and the images of the second set are the generators of \( R^{(n+1)(n-2)/2} \).

Each surface \( F_2 \) of the system meets the ruled manifold \( R^{(n+1)(n-2)/2} \) in \((n+1)(n-2)/2\) lines and contains \( n+1 \) other lines.

Hence the \( \infty^n \) sections of the \( F_2 \) of the system are mapped upon a plane by the system of curves of order \( n \) through \( n(n+1)/2 \) points.

For an arbitrary plane, the lines on the image surface \( F_2 \) are mutually skew. A line in the image \( (\lambda, \mu, \nu) \) plane is the image of a normal curve \( C_n \) of order \( n \) in \( S_n \). If the image line passes through a base point, the \( C_n \) consists of a normal \( C_{n-1} \) of order \( n-1 \) in \( S_{n-1} \) and one of the \( n(n+1)/2 \) lines intersecting it in one point. The residual section in the \( S_{n-1} \) is a curve \( C_{n(n-3)/2} \) of order \( n(n-3)/2 \) having for an image in the \( (\lambda, \mu, \nu) \) plane a \( C'_{n-1} \) through the other \( n(n-1)/2 \) points; it is of genus \((n-2)(n-3)/2\). Through this \( C_{n(n-3)/2} \) there pass \( \infty^1 \) primes, each containing a \( C_{n-1} \), hence the \( C_{n(n-3)/2} \) lies in an \([n-2]\)-dimensional space \( S_{n-2} \), and each \( S_{n-1} \) through it meets \( F_2 \) in a \( C_{n-1} \) which intersects the line not secant to \( C_{n(n-3)/2} \) and also the latter curve in \( n-1 \) points.

Among the lines of the pencil \( A_i \), images of curves \( C_{n-1} \) on \( F_2 \), there is one through another base point \( A_k \). The curve on the surface \( F_2 \) having this line for image consists of two base lines and a curve \( C_{n-2} \) of order \( n-2 \), hence lying in an \( S_{n-2} \). Since the residual plane curve passes through \( A_k \) and the composite curve is the image of a prime section of \( F_2 \), it follows that the residual curve on \( F_2 \) is tangent to the \( S_{n-3} \) passing through \( n-1 \) curves \( C_{n-2} \), and the space of each \( C_{n-2} \) is tangent to two residual curves. This residual curve \( C_{(n^2-3n+4)/2} \) of order \((n^2-3n+4)/2\) and of genus \((n-3)(n-4)/2\) intersects \( C_{n-2} \) in \( n-2 \) points. There are therefore \( n(n-1)/2 \) curves \( C_{n-2} \) of \( S_{n-2} \) lying on \( F_2 \), intersecting in pairs in not more than one point; if their image lines meet in the same base points, the associated \( C_{n-2} \) do not intersect. The curves \( C_{(n^2-3n+4)/2} \) intersect in pairs in \((n-2)(n-3)/2\) points apart from the base points.
Any two $F_2$ of the system do not have any points in common. The surfaces $F_2$ do not contain rational curves of order less than $n - 2$, apart from the base lines.

For $n = 4$, an $S_3$ meets every pencil of the quartic primals $|\psi_4|$ in a pencil of quartic surfaces, each containing the intersection of the basic two-dimensional surface $F_2$ of order six and $S_3$, a curve $C_6$ of order six and of genus three. Hence each quartic surface is invariant under an infinite discontinuous group of Cremona transformations.*

Similarly, for $n = 5$, an $S_4$ meets a net of quintic primals in a net of quintic surfaces, each passing through the intersection of the basic two-dimensional surface $F_2$ of order ten (that is, in ten points) and an $S_4$, a curve $C_4$ of order ten and of genus six. An $S_4$ containing a normal $C_4$ on $F_2$ meets it in a residual $C_6$ of order six and of genus three, which intersects $C_4$ in four points. But through this residual $C_6$ pass $\infty^1 |S_4|$, each meeting $F_2$ in a normal $C_4$ of $S_4$; hence the $C_6$ lies in the base of the pencil $|S_4|$, and the quartic surfaces through it have the same property as that mentioned in the case $n = 4$.

This property is true for prime sections of pencils of primals in $S_n$, since their equations can always be written in determinantal form.

Any $S_{n-2}$ through one of the bases, for example, $p = q = r = 0$, meets each homologous $S_{n-2}$ of the other projective systems in $n - 1$ spaces $S_{n-4}$. The pencil of $|S_{n-1}|$ through the homologous $S_{n-2}$ of the base $p = q = r = 0$ meets the base in a pencil $|S_{n-3}|$. Similarly for each of the other systems. There are then in the arbitrary $S_{n-2}$ through the base $n - 1$ spaces $S_{n-4}$, axes of projective pencils of $|S_{n-3}|$, each of the form $a + \lambda b = 0$. The condition that these $n - 1$ spaces are concurrent is expressed by an equation of order $n - 1$ in $\lambda$. Hence in the given $S_{n-2}$ there are $n - 1$ variable points of $F_2$. Since an $S_{n-2}$ meets $F_2$ in $(n + 1)n/2$ points, it follows that each base meets $F_2$ in $(n - 1)(n - 2)/2$ points in addition to a base curve.

(c) If $(x)$ describes an $S_3$, the equations may be written in the form $x_i' = f_i(\lambda, \mu, \nu, \rho)$, each $f_i$ being a polynomial of order $n$. These three-dimensional varieties are presented by their prime sections, which are expressed in terms of surfaces of order $n$, all passing through a surface of order $(n - 3)\cdot n C_3 \cdot (n + 1)/3$.

* Snyder and Sharpe, Transactions of this Society, vol. 16 (1915), pp. 62-70.
If the $n+1$ projective systems of lines, planes, $\cdots$, primes with point vertices $A_0$, $\cdots$, $A_n$, are considered arbitrarily in $S_n$, the locus of intersections of corresponding primes of the system is a variety $M$ of order $(n+1)n/2$ and of dimensionality $n-2$. The locus of intersections of corresponding primes of $n$ of the systems, omitting $A_0$, is a determinantal primal of order $n$, $\Phi_i$, passing through all the vertices of the system except $A_0$ and containing $M$. In the same way primals $\Phi_i$ are obtained, $(i=1, 2, \cdots, n)$, each containing $M$ and all the vertices except $A_i$. The $n+1$ primals are linearly independent.

Similarly, for the loci of intersections of corresponding primes of $n-1$, $n-2$, $\cdots$, $3$, $2$ of the systems.

The synthetic argument made by Todd* for $n=4$ can be extensively generalized to apply for $n$ general. There are $n$ degrees of freedom to account for the $\infty^* \text{ primes of } S_n$. There is a base curve of order $(n+1)n/2$, the intersection of a manifold $M$ of order $(n+1)n/2$ and an $S_3$. The genus of this curve is determined by the postulation. A surface of order $n$ meets a curve of order $m$ and of genus $p$ in $mn-p+1$ points in order to pass through the curve. The postulation of the surface is $[(n+3)(n+2)(n+1)/6]-1-n$, hence

$\frac{n^2(n+1)}{2} - p + 1 = \frac{(n+3)(n+2)(n+1)}{6} - 1 - n,$

from which $p = (n-1)(n-2)(2n+3)/6$. The projective systems of primes through the three spaces $S_3'$ as bases generate a three-dimensional rational variety of $S_n$ mapped on $S_3$ by means of surfaces of order $n$ passing through a curve of order $(n+1)n/2$ and of genus $(n-1)(n-2)(2n+3)/6$. These varieties have $(n+1)n(n-1)/6$ singular points, mapped by the multisecants of the base curve.

7. The Base Elements of the Veneroni Transformation. The $n+1$ base $S_{n-2}$ form part of the base $[n-2]$-dimensional manifold $M$ of order $(n+1)n/2$, defined by the matrix $||a_{ik}||$; the residual is a ruled variety $R$ of order $(n+1)(n-2)/2$ formed by the lines which meet all the $n+1$ base $S_{n-2}$.

Each normal $C_n$ image of a line meets each base $S_{n-2}$ in $n-2$ points and does not intersect the ruled variety.

The images of planes intersect $R$ in $(n+1)(n-2)/2$ lines. The plane meets each base $S_{n-2}$ in a point, the image of which is a line meeting $n$ of the base $S'_n$ and lying on $F_2$. Each base $S_{n-2}$ meets $R$ in a manifold of dimensionality $n-3$ and of order $n-1$. For $n = 4$, the two-dimensional variety of order 5 has an infinite number of plane elliptic cubic curves, but the corresponding property is not true for larger values of $n$ although the intersections of each base $S_{n-2}$ and $R$ are birationally equivalent.

CORNELL UNIVERSITY AND
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ON THE CHARACTERISTIC ROOTS OF
MATRIC POLYNOMIALS*

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1. Introduction. Unless otherwise stated, all matrices and polynomials are assumed to have coefficients in an arbitrary algebraically closed field $K$.

Let $A$ and $B$ denote square matrices of order $n$. If the characteristic roots of every polynomial $f(A, B)$ are all of the form $f(\lambda, \mu)$, where $\lambda$ and $\mu$ are characteristic roots of $A$ and $B$, respectively, then in accordance with a notation to be introduced below, we shall say that the matrices $A$, $B$ have property $I_n$. By a theorem of Frobenius,† the matrices $A$, $B$ have this property if they are commutative, but this is by no means a necessary condition. The study of pairs of matrices having property $I_n$ has been the subject of papers by Bruton, Ingraham, and Roth.‡ However, in no case have conditions been obtained which are both necessary and sufficient for the existence of this property.

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